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Principles of Distributed Computing
Exercise 13: Sample Solution

1 Flow labeling schemes

Question 1 Check that R_k is reflexive, symmetric and transitive.

- reflexive: $\text{flow}(x, x) = \infty$
- symmetric: the graph is undirected, $\text{flow}(x, y) = \text{flow}(y, x)$
- transitive: consider a path $p = (v_1, v_2, \ldots, v_{m_p})$ from x to y in which $v_1 = x$ and $v_{m_p} = y$ and a path $p' = (v'_1, v'_2, \ldots, v'_{m_{p'}})$ from y to z in which $v'_1 = y$ and $v'_{m_{p'}} = z$. Let i be the largest subscript in p' such that $v'_i \in p$. It is easy to check there is a path $x - -v'_i - -z$ where $x - -v'_i$ is a part of p and $v'_i - -z$ is a part of p'.

C_{k+1} is a refinement of C_k.

Question 2 Add the depth of each vertex into the label, and the depth of the tree is smaller than m.
See that

$$\text{flow}_G(v, w) = \text{SepLevel}_T(t(v), t(w)) + 1. \quad (1)$$

The depth of T_G cannot exceed $n\hat{\omega}$ and every level at most has n nodes, hence the total number of nodes in T_G is $O(n^2\hat{\omega})$.

Question 3 Cancel all nodes of degree 2 in T_G, and add appropriate edge weights ($\tilde{T}_G$).

Now, define $\text{SepLevel}_T(x, y)$ as the weighted depth of $z = lca(x, y)$, i.e., its weighted distance from the root. We can also extend SepLevel labeling schemes to weighted trees. For $\tilde{n}$-node trees with maximum weight $\tilde{\omega}$, the labeling size is $O(\log \tilde{n} \log \tilde{\omega} + \log^2 \tilde{n})$.

It is also easy to verify that for two nodes x, y in G, the separation level of the leaves $t(x)$ and $t(y)$ associated with x and y in the tree $\tilde{T}_G$ is still related to the flow between the two vertices, similar to Eq. (1).

Finally, note that as $\tilde{T}_G$ has exactly n leaves, and every non-leaf node in it has at least two children, the total number of nodes in $\tilde{T}_G$ is $\tilde{n} \leq 2n - 1$. Moreover, the maximum edge weight in $\tilde{T}_G$ is $\tilde{\omega} \leq n\hat{\omega}$.

For more details, one can check the paper [1] (Section 2).

References

[1] Katz, Michal, et al., *Labeling schemes for flow and connectivity*, SIAM Journal on Computing 34.1 (2004): 23-40.