

## Exercise 7

Lecturer: Mohsen Ghaffari

## 1 Lower Bound for Locally-Minimal Coloring

For a graph  $G = (V, E)$ , a coloring  $\phi : V \rightarrow \{1, 2, \dots, Q\}$  is called *locally-minimal* if it is a proper coloring, meaning that no two adjacent vertices  $v$  and  $u$  have  $\phi(v) = \phi(u)$ , and moreover, for each node  $v$  colored with color  $q = \phi(v) \in \{1, 2, \dots, Q\}$ , all colors  $1$  to  $q - 1$  are used in the neighborhood of  $v$ . That is, for each  $i \in \{1, \dots, q - 1\}$ , there exists a neighbor  $u$  of  $v$  such that  $\phi(u) = i$ .

### Exercise

- (1a) In the 5<sup>th</sup> lecture, we saw a  $O(\Delta \log \Delta + \log^* n)$ -round algorithm for computing a  $(\Delta + 1)$ -vertex-coloring in any  $n$ -node graph with maximum degree  $\Delta$ . Use this algorithm as a black box to compute a *locally-minimal coloring* in  $O(\Delta \log \Delta + \log^* n)$  rounds, in an  $n$ -node graph with maximum degree  $\Delta$ .

In the remainder of this exercise, we prove a lower bound of  $\Omega(\log n / \log \log n)$  on the round complexity of computing a *locally-minimal coloring*, for some graphs. We note that these graphs have maximum degree  $\Delta = \Omega(\log n)$  and hence, this lower bound poses no contradiction with (1a).

For the lower bound, we will use a classic graph-theoretic result of Erdős [Erd59]. Recall that the girth of a graph is the length of its shortest cycle, and the chromatic number of a graph is the smallest number of colors required in any proper coloring of the graph.

**Theorem 1 (Erdős [Erd59])** *For any sufficiently large  $n$ , there exists an  $n$ -node graph  $G_n^*$  with girth  $g(G_n^*) \geq \frac{\log n}{4 \log \log n}$  and chromatic number  $\chi(G_n^*) \geq \frac{\log n}{4 \log \log n}$ .*

### Exercise

- (1b) Prove that in any locally-minimal coloring  $\phi : V \rightarrow \{1, 2, \dots, Q\}$  of a tree  $T = (V, E)$  with diameter  $d$  — i.e., where the distance between any two nodes is at most  $d$  — no node  $v$  can receive a color  $\phi(v) > d + 1$ .
- (1c) Suppose towards contradiction that there exists a deterministic algorithm  $\mathcal{A}$  that computes a locally-minimal coloring of any  $n$ -node graph in at most  $\frac{\log n}{8 \log \log n} - 1$  rounds. Prove that when we run  $\mathcal{A}$  on the graph  $G_n^*$ , it produces a (locally-minimal) coloring with at most  $Q = \frac{\log n}{4 \log \log n} - 1$  colors. For this, you should use part (1b) and the fact that  $G_n^*$  has girth  $g(G_n^*) \geq \frac{\log n}{4 \log \log n}$ .
- (1d) Conclude that any locally-minimal coloring algorithm needs at least  $\frac{\log n}{8 \log \log n}$  rounds on some  $n$ -node graph.

## References

[Erd59] Paul Erdős. Graph theory and probability. *Canada J. Math*, 11:34G38, 1959.