Lecture 9

Exercise 1

The algorithm for MIS follows in the footsteps of the $(\Delta+1)$-coloring from the lecture. Notice that since a strong decomposition is also a weak one, it is enough to provide a solution to the weak case.

Let $G_1, \ldots, G_{\mathcal{C}}$ be the vertex disjoint graphs promised by the $(\mathcal{C}, \mathcal{D})$ weak network decomposition. First, we solve MIS in every cluster $X_1, \ldots, X_\ell$ of G_1 by the trivial algorithm that collects all information to a single node and solves the problem centrally. Since the diameter of G_1 is bounded by $\mathcal{D}$, the time needed is $O(\mathcal{D})$. Furthermore, since all the clusters in G_1 are mutually non-adjacent, the MIS of a cluster X_i does not break the MIS of X_j for any $i \neq j$.

Now, consider the set of nodes M that are in the MIS. These nodes and their neighbors are "happy" in the sense that either they are in the MIS or their neighbor is in the MIS. Therefore, we can just ignore these nodes for here on. This can be thought of as simply removing them from the graph. Notice that the nodes that remain are not adjacent to any MIS nodes and every node in G_1 is removed.

Then, we perform the same process iteratively to $G_2, \ldots, G_{\mathcal{C}}$ and in total, get the runtime of $O(\mathcal{C}\mathcal{D})$.

Exercise 2

Assume for contradiction that a $(o(\log n), o(\log n/\log\log n))$ weak network decomposition exists in a graph G with girth $s = \Omega(\log n/\log\log n)$ and chromatic number $\Omega(\log n)$. Since the diameter of any cluster of the network decomposition is in $o(\log n/\log\log n)$, we can safely assume that the diameter of any cluster in any subgraph is less than $s/3$.

Consider now the subgraph G_1 with clusters $X_1, \ldots, X_\ell$. By the definition of girth, no cluster can have a cycle of length less than s. Assume then again for contradiction, that there is a cluster X_j a cycle and let u and v be nodes along this cycle with maximum distance, i.e., distance at least $s/2 - 1 > s/3$. Now, since there are no cycles of length less than s, the shortest path from u to v has to be of length strictly larger than $s/3$, which is a contradiction to the fact that the diameter of X_j is $s/3$.

In other words, no cluster can contain cycles which implies that every cluster has to be a tree. Every tree can be colored with 2 colors. Furthermore, since all clusters of any G_i are mutually non-adjacent, we can color the whole graph G_i with two colors. Using a different color palette of two colors for every G_i yields a coloring with $2 \cdot o(\log n) = o(\log n)$ colors, which is a contradiction to the fact that the chromatic number is in $\Omega(\log n)$.

Therefore, a $(o(\log n), o(\log n/\log\log n))$ weak network decomposition cannot exist.

Exercise 3

The goal in this exercise is to show that an $O(\log^2 n)$-diameter ordering exists. To show this, it suffices to find an ordering for any given graph G with an $(O(\log n), O(\log n))$ weak network decomposition.

The first observation is that if we look only at graph G_1 of a network decomposition, we can choose labels arbitrarily and get a $O(\log n)$-diameter ordering within this subgraph (G_1). Let us assume that the number of nodes in G_i is k_i and let us use labels $1, \ldots, k_1$ to label the nodes in G_1 arbitrarily. Then, we continue the labeling process by labeling graph G_2 arbitrarily with labels $k_1 + 1, \ldots, k_2$. The crucial observation here is that since every label of G_1 is strictly smaller than any label in G_2, there cannot be a monotonically increasing path $u_1, \ldots, u_\ell$, for any $\ell > 1$, of labels such that u_i is a node of G_2 and u_{i+1} is node of G_1. Intuitively, this means that no monotonically increasing path can lead back to a subgraph with a smaller index.

Assume now that the labeling process is performed through all graphs $G_1, \ldots, G_{\mathcal{C}}$. Let $P = v_1, \ldots, v_r$ be any path in G with monotonically increasing labels. Notice that r can be large, up to n. However, since it cannot be the case that v_i belongs to G_j and v_{i+1} to $G_{j'}$ for any $j' < j$, the path P consists of at most $\mathcal{C}$ subpaths where every subpath belongs to exactly one subgraph. It follows from the definition

of the weak network decomposition that the distance (in graph G) of any pair of nodes $v_i, v_{j>i}$ of any subpath is bounded from above by $O(\log n)$. Since there are $\mathcal{C} = O(\log n)$ subpaths, the distance of any pair of nodes in the whole path is bounded from above by $O(\log^2 n)$.