| | |
|---|---|
| **Principles of Distributed Computing** | 04/25, 2018 |

# Exercise 9

*Lecturer: Mohsen Ghaffari*

## Exercise 1

The algorithm for MIS follows in the footsteps of the $(\Delta+1)$-coloring from the lecture. Notice that since a strong decomposition is also a weak one, it is enough to provide a solution to the weak case.

Let $G_1, \ldots, G_{\mathcal{C}}$ be the vertex disjoint graphs promised by the $(\mathcal{C}, \mathcal{D})$ weak network decomposition. First, we solve MIS in every cluster $X_1, \ldots, X_\ell$ of $G_1$ by the trivial algorithm that collects all information to a single node and solves the problem centrally. Since the diameter of $G_1$ is bounded by $\mathcal{D}$, the time needed is $O(\mathcal{D})$. Furthermore, since all the clusters in $G_1$ are mutually non-adjacent, the MIS of a cluster $X_i$ does not break the MIS of $X_j$ for any $i \neq j$.

Now, consider the set of nodes $M$ that are in the MIS. These nodes and their neighbors are "happy" in the sense that either they are in the MIS or their neighbor is in the MIS. Therefore, we can just ignore these nodes from here on. This can be thought of as simply removing them from the graph. Notice that the nodes that remain are not adjacent to any MIS nodes and every node in $G_1$ is removed.

Then, we perform the same process iteratively to $G_2, \ldots, G_{\mathcal{C}}$ and in total, get the runtime of $O(\mathcal{C}\mathcal{D})$.

## Exercise 2

Assume for contradiction that a $(o(\log n), o(\log n/\log\log n))$ weak network decomposition exists in a graph $G$ with girth $s = \Omega(\log n/\log\log n)$ and chromatic number $\Omega(\log n)$. Since the diameter of any cluster of the network decomposition is in $o(\log n/\log\log n)$, we can safely assume that the diameter of any cluster in any subgraph is less than $s/3$.

Consider now the subgraph $G_1$ with clusters $X_1, \ldots, X_\ell$. By the definition of girth, no cluster can have a cycle of length less than $s$. Assume then again for contradiction, that there is a cluster $X_j$ containing a cycle and let $u$ and $v$ be nodes on this cycle with maximum distance along the cycle, i.e., distance at least $s/2 - 1 > s/3$. Now, since there are no cycles of length less than $s$, the shortest path from $u$ to $v$ has to be of length strictly larger than $s/3$, which is a contradiction to the fact that the diameter of $X_j$ is $s/3$.

In other words, no cluster can contain cycles which implies that every cluster has to be a tree. Every tree can be colored with 2 colors. Furthermore, since all clusters of any $G_i$ are mutually non-adjacent, we can color the whole graph $G_i$ with two colors. Using a different color palette of two colors for every $G_i$ yields a coloring with $2 \cdot o(\log n) = o(\log n)$ colors, which is a contradiction to the fact that the chromatic number is in $\Omega(\log n)$.

Therefore, a $(o(\log n), o(\log n/\log\log n))$ weak network decomposition cannot exist.

## Exercise 3

The goal in this exercise is to show that an $O(\log^2 n)$-diameter ordering exists, for any given graph $G$. To show this, we use the fact that any graph has a $(O(\log n), O(\log n))$ weak network decomposition.

The first observation is that if we look only at graph $G_1$ of a network decomposition, we can choose labels arbitrarily and get a $O(\log n)$-diameter ordering within this subgraph ($G_1$). Let us assume that the number of nodes in $G_i$ is $k_i$ and let us use labels $1, \ldots, k_1$ to label the nodes in $G_1$ arbitrarily. Then, we continue the labeling process by labeling graph $G_2$ arbitrarily with labels $k_1 + 1, \ldots, k_1 + k_2$. The crucial observation here is that since every label of $G_1$ is strictly smaller than any label in $G_2$, there cannot be a monotonically increasing path $u_1, \ldots, u_\ell$, for any $\ell > 1$, of labels such that $u_i$ is a node of $G_2$ and $u_{i+1}$ is a node of $G_1$. Intuitively, this means that no monotonically increasing path can lead back to a subgraph with a smaller index.

Assume now that the labeling process is performed through all graphs $G_1, \ldots, G_{\mathcal{C}}$. Let $P = v_1, \ldots, v_r$ be any path in $G$ with monotonically increasing labels. Notice that $r$ can be large, up to $n$. However, since it cannot be the case that $v_i$ belongs to $G_j$ and $v_{i+1}$ to $G_{j'}$ for any $j' < j$, the path $P$ consists of at most $\mathcal{C}$ subpaths where every subpath belongs to exactly one subgraph. It follows from the definition of the weak network decomposition that the distance (in graph $G$) of any pair of nodes $v_i, v_{j>i}$ of any subpath is bounded from above by $O(\log n)$. Since there are $\mathcal{C} = O(\log n)$ subpaths, the distance of any pair of nodes in the whole path is bounded from above by $O(\log^2 n)$.