HOW EFFECTIVE CAN SIMPLE ORDINAL PEER GRADING BE?

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MASSIVE OPEN ONLINE COURSES

edX

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coursera

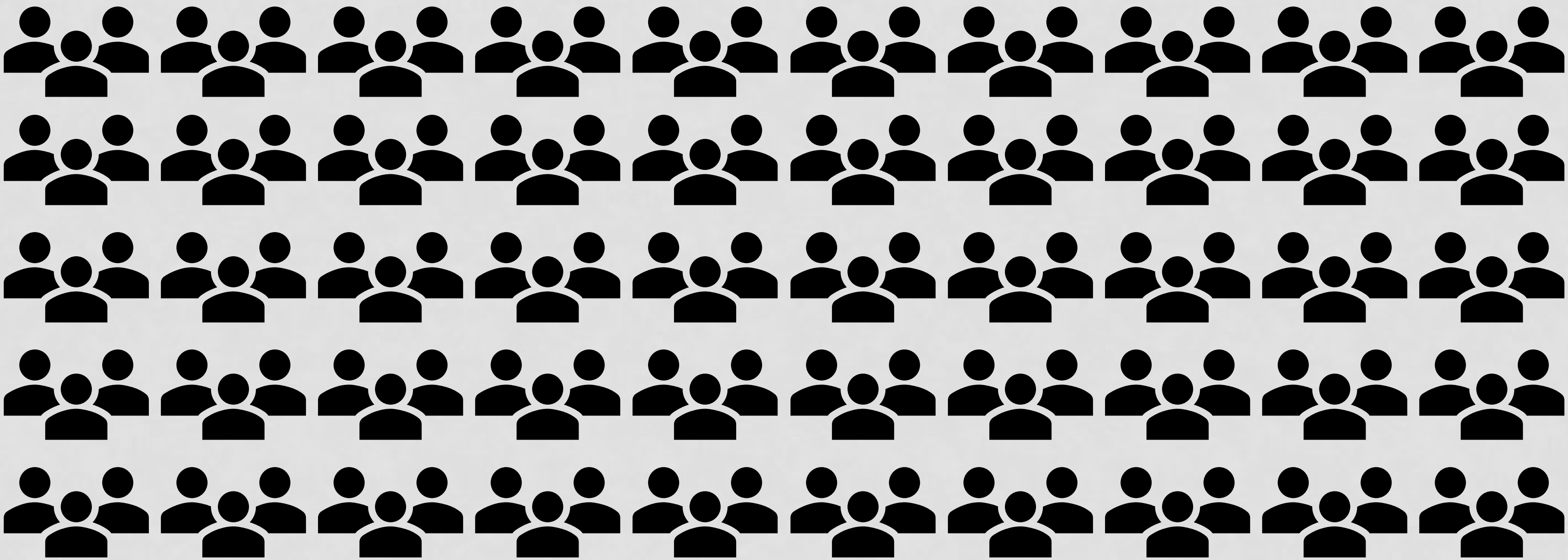
kadenze

UDACITY

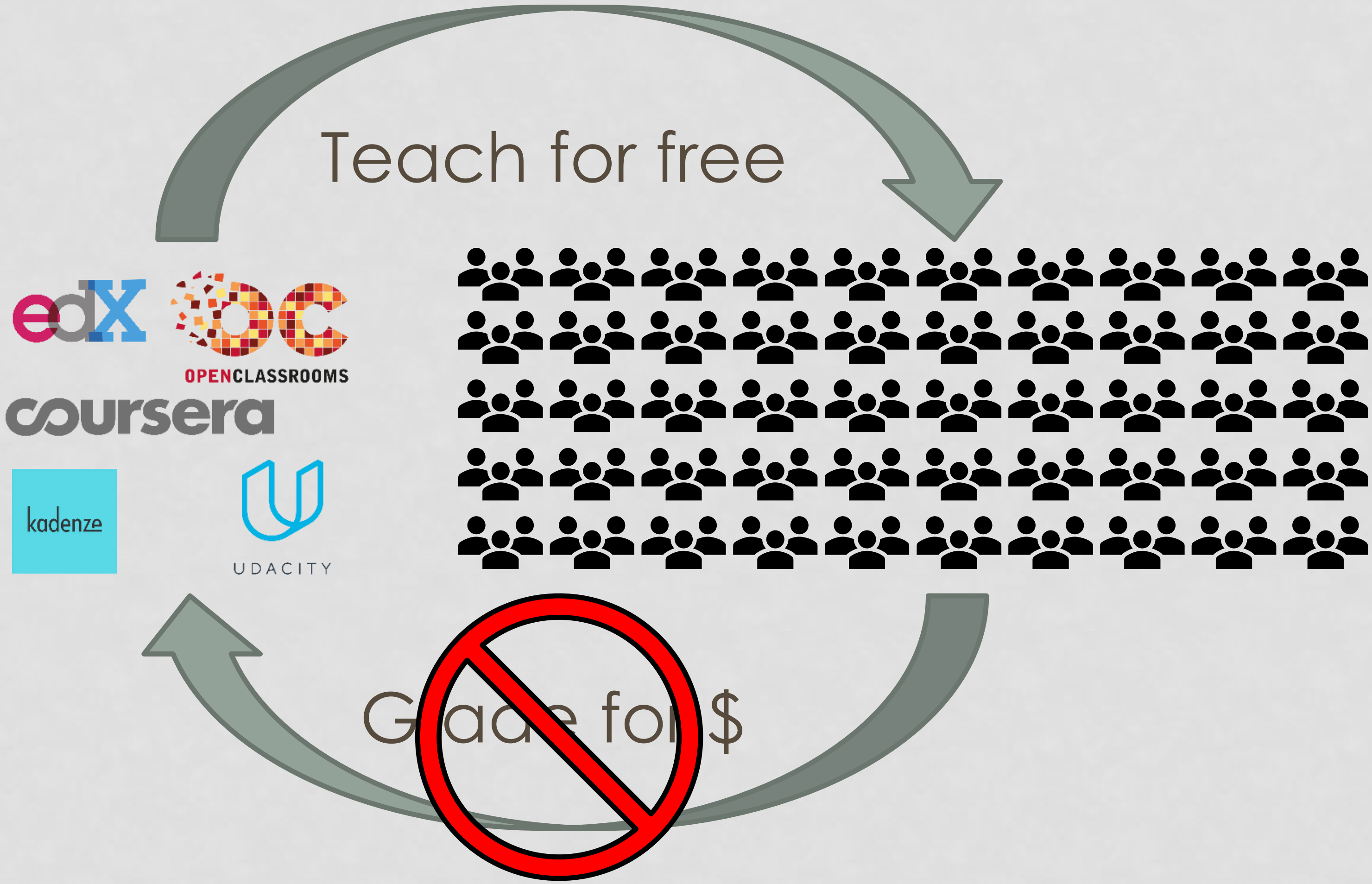
BUSINESS MODEL

PROBLEM

Up to 50 000 students per course

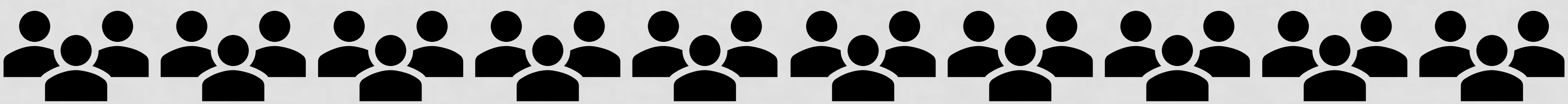


BUSINESS MODEL VS PROBLEM



IDEA – PEER GRADING

Outsource grading to students



HOW TO GRADE

- Cardinal grading (absolute grading)

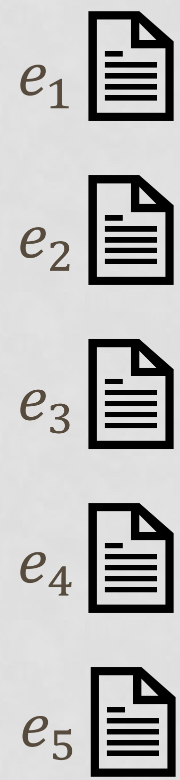
	Topic	Max. Points	Points	Signature
1	ML & Bayesian inference	20		
2	Kernels	20		
3	Neural Networks	20		
4	Gaussian processes	20		
5	Unsupervised learning	20		
Total		100		

Grade: . . . .

Source: Machine Learning Exam 2015 (ETHZ)

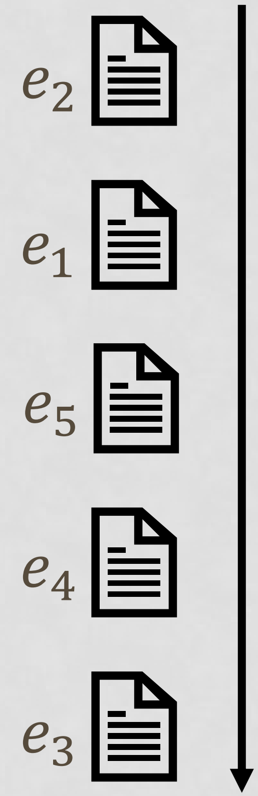
HOW TO GRADE

- Ordinal grading (sorting)



HOW TO GRADE

- Ordinal grading (sorting)



HOW TO GRADE

- Cardinal grading (absolute grading)
  - *Assign low grades → improve own performance,*
  - *Lack of experience.*

- Ordinal grading (sorting)
  - *Free from incentive to under-grade,*
  - *Requires less grading experience.*

HOW TO GRADE

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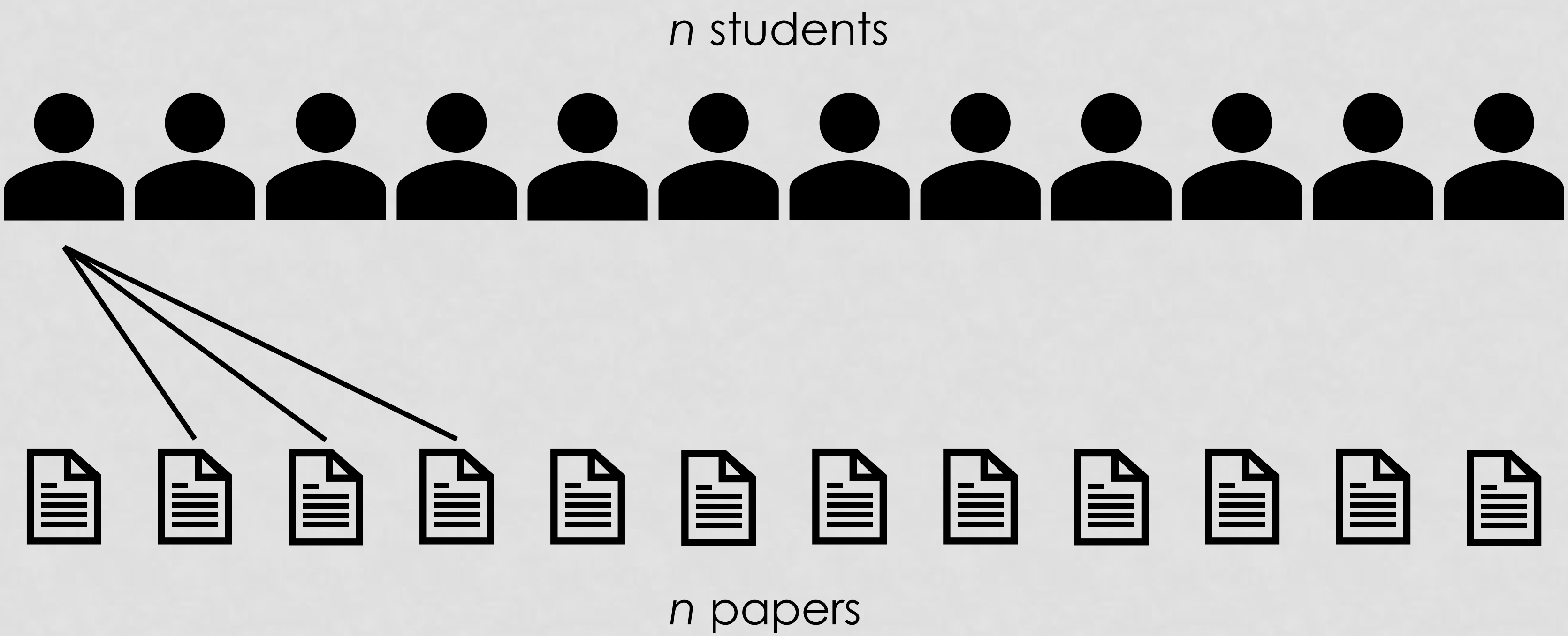
SETTING

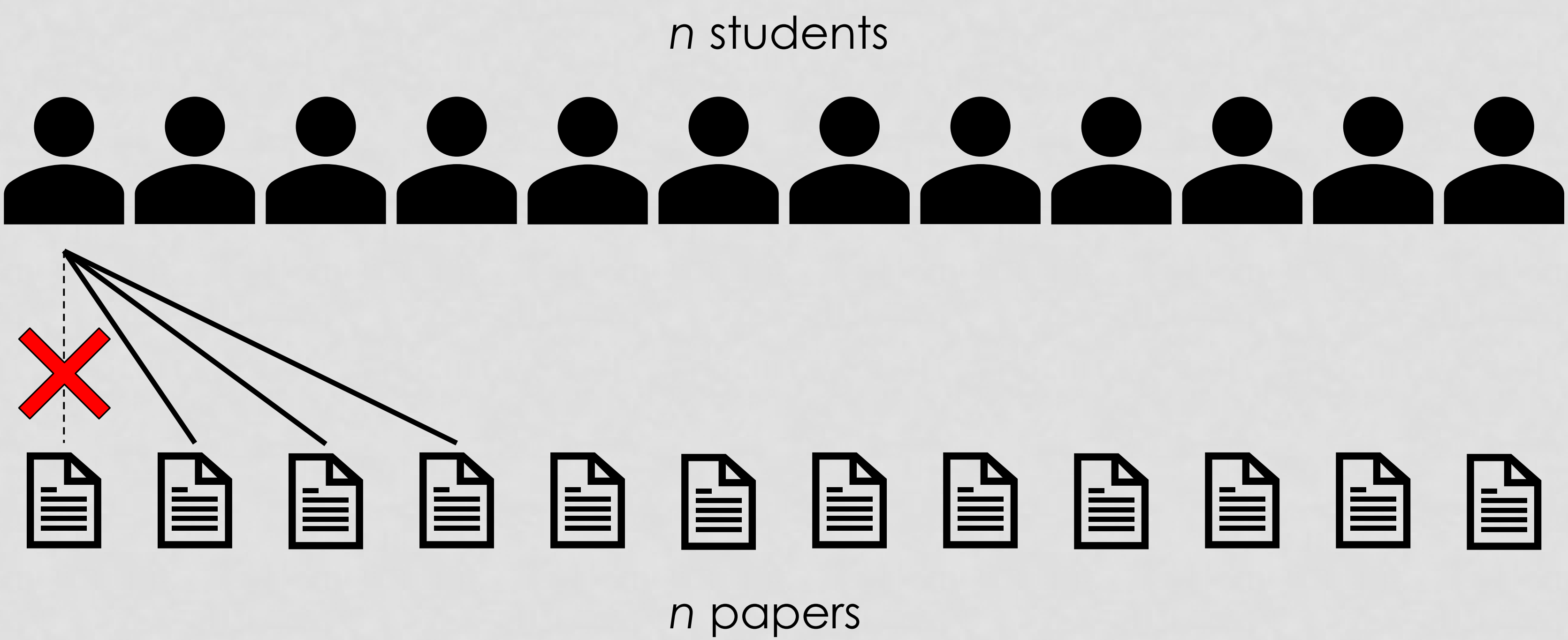
n students

n papers

BUNDLES OF k EXAMS



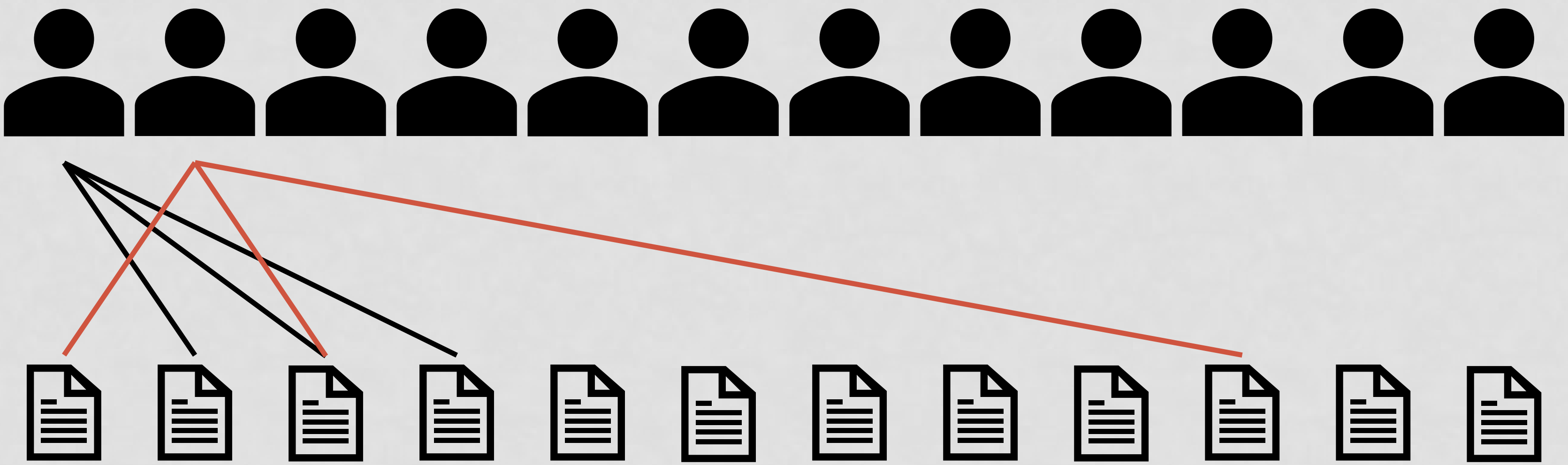
BUNDLES OF *k* EXAMS



Student cannot grade his own paper

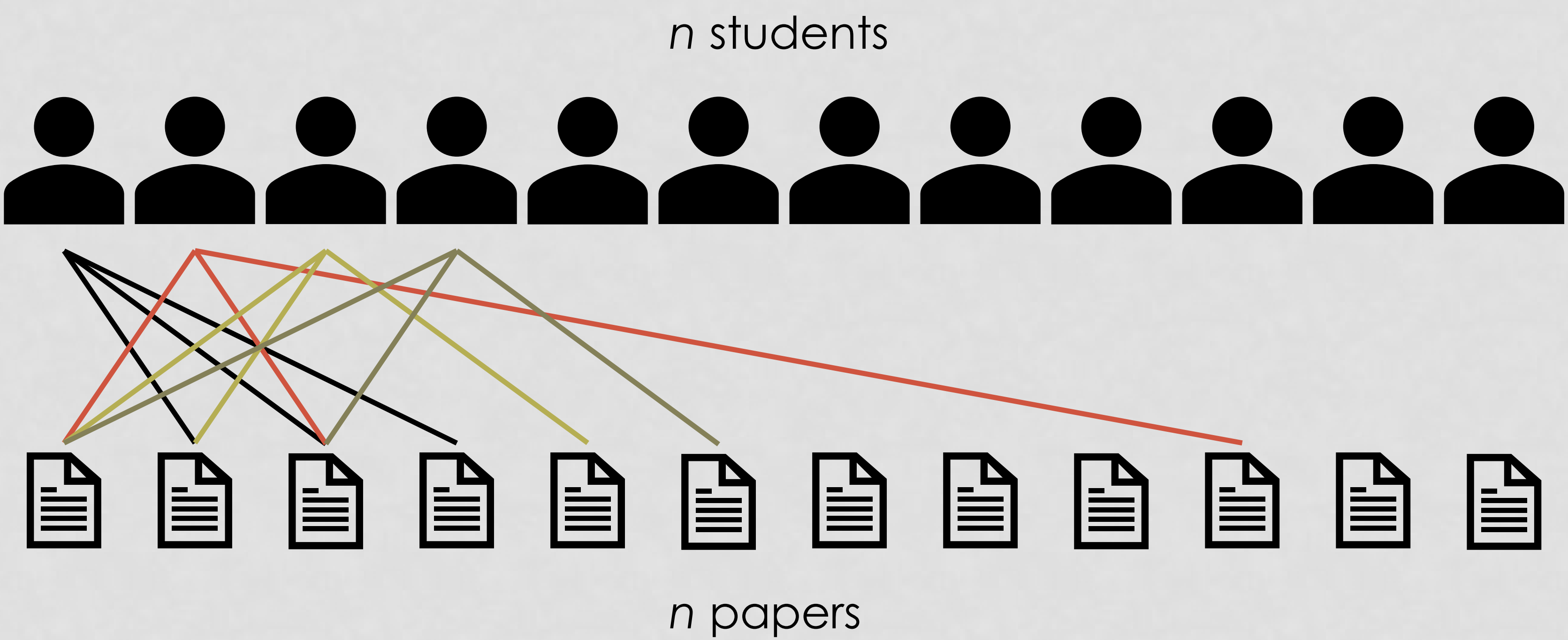
SETTING

n students



n papers

SETTING



SETTING

n students

n papers

SETTING

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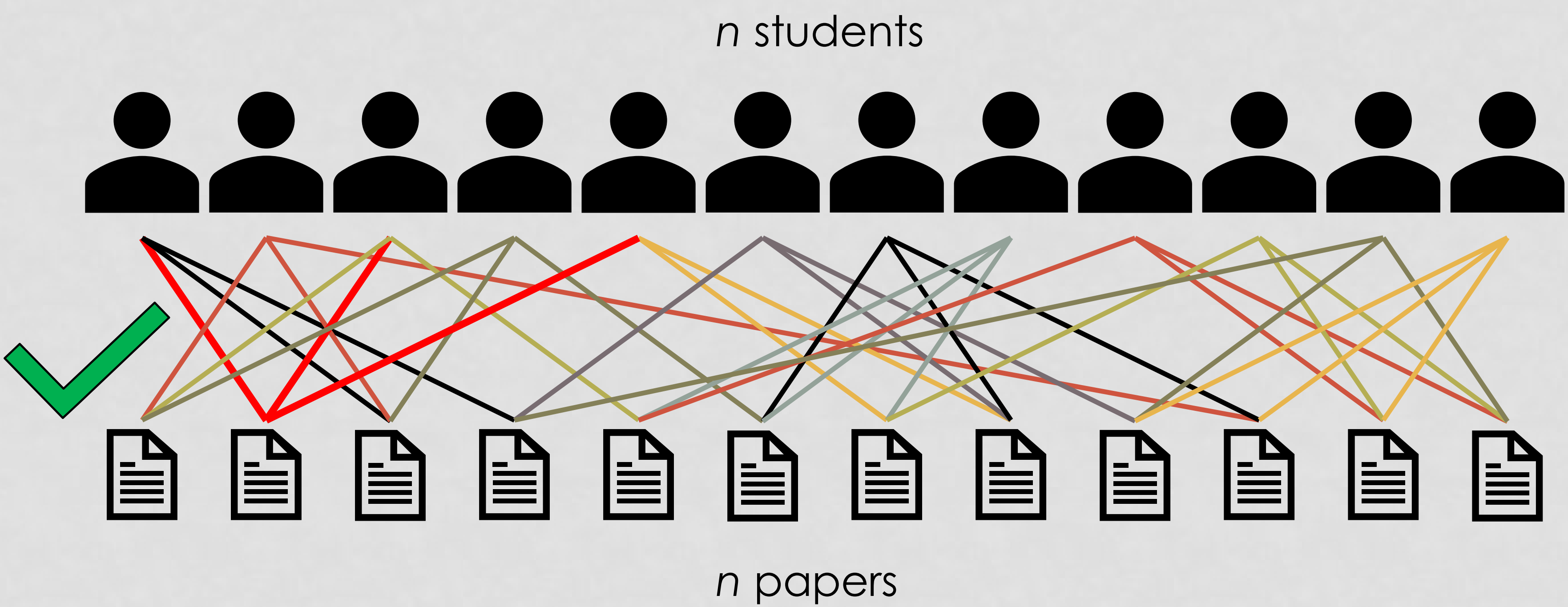
n papers

SETTING

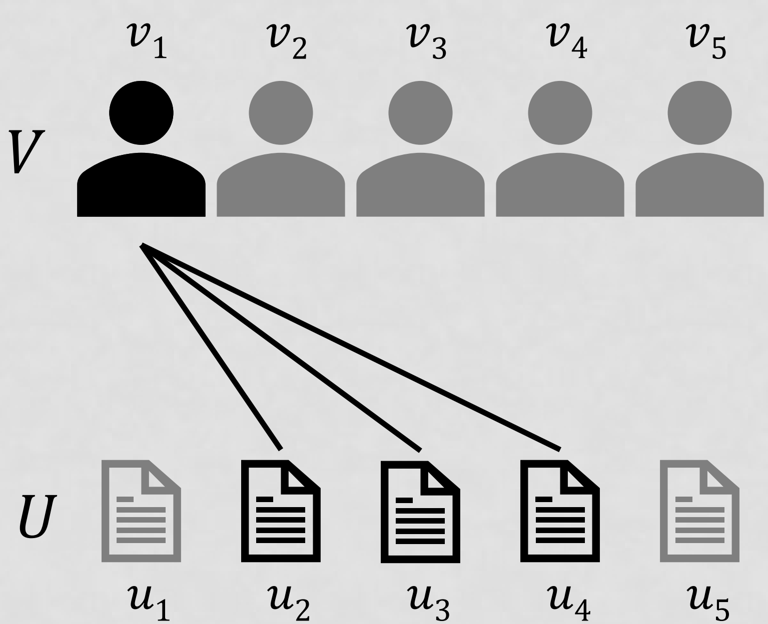


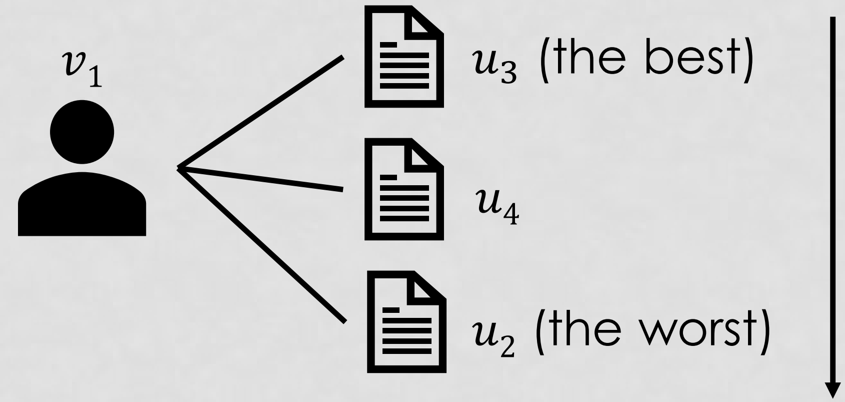
SETTING

n students

n papers

SINGLE GRADER





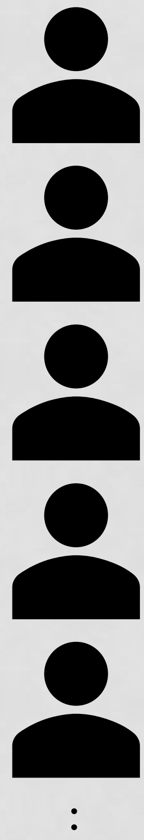
$$v_1 = \begin{bmatrix} u_3 \\ u_4 \\ u_2 \end{bmatrix} \downarrow$$

Is this ordering correct?

MODELLING STUDENTS' GRADING BEHAVIOUR

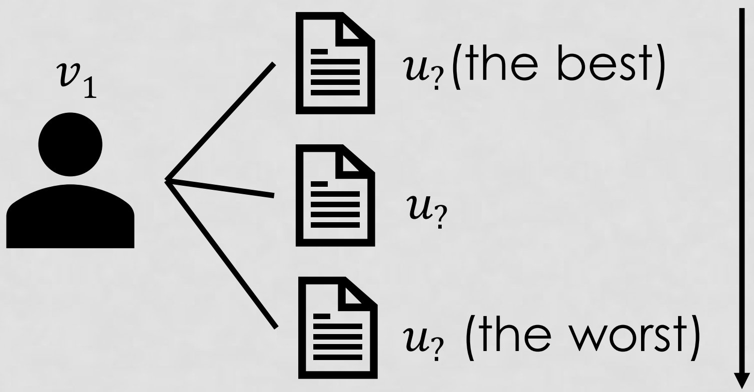
GROUND TRUTH

1.
2.
3.
4.
5.



GROUND TRUTH

SINGLE STUDENT'S RANKING



$$v_1 = \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} \qquad v_1 = \begin{bmatrix} u_2 \\ u_4 \\ u_3 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} u_3 \\ u_2 \\ u_4 \end{bmatrix} \qquad v_1 = \begin{bmatrix} u_3 \\ u_4 \\ u_2 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} u_4 \\ u_3 \\ u_2 \end{bmatrix} \qquad v_1 = \begin{bmatrix} u_4 \\ u_2 \\ u_3 \end{bmatrix}$$

Which ordering is correct?

AGGREGATE INFORMATION

$$P = \begin{bmatrix} p_{1,1} & \cdots & p_{1,k} \\ \vdots & \ddots & \vdots \\ p_{k,1} & \cdots & p_{k,k} \end{bmatrix}$$

EXAMPLE ($k = 3$)

Perfect Graders

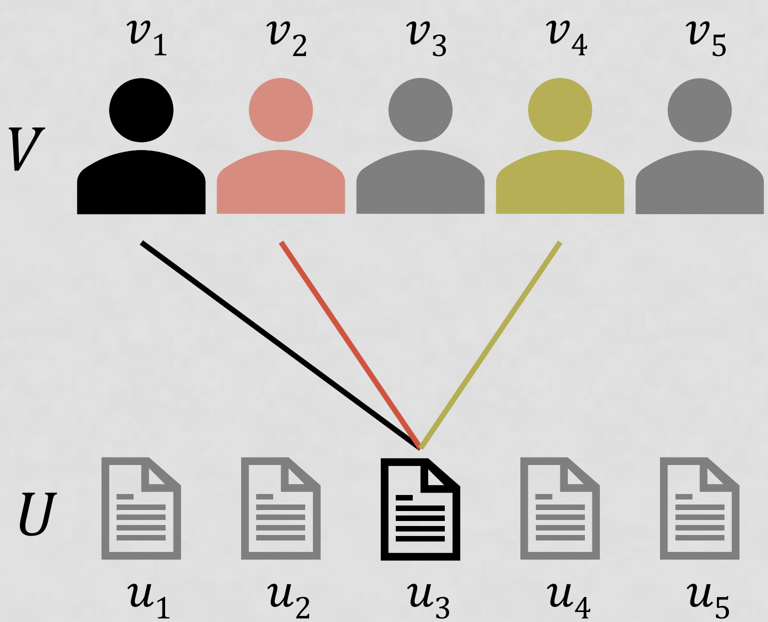
$$P = \begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

EXAMPLE ($k = 3$)

Not Perfect Graders

$$P = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

SINGLE EXAM



$$v_1 = \begin{bmatrix} \mathbf{u_3} \\ u_4 \\ u_2 \end{bmatrix} \quad v_2 = \begin{bmatrix} \mathbf{u_3} \\ u_5 \\ u_1 \end{bmatrix} \quad v_4 = \begin{bmatrix} u_7 \\ \mathbf{u_3} \\ u_1 \end{bmatrix}$$

Type – grading result of exam paper

$$\sigma_{u_3} = (1,1,2)$$

AGGREGATION RULE - BORDA

Extract types

u_1 u_2 u_3 u_4 u_5

$$\sigma_{u_1} = (1,1,1)$$

$$\sigma_{u_2} = (1,2,3)$$

$$\sigma_{u_3} = (1,1,2)$$

$$\sigma_{u_4} = (2,2,3)$$

$$\sigma_{u_5} = (1,1,2)$$

AGGREGATION RULE - BORDA

Compute Borda Score

u_1 u_2 u_3 u_4 u_5

$$\sigma_{u_1} = (1,1,1) \rightarrow B(\sigma_{u_1}) = 3 + 3 + 3 = 9$$

$$\sigma_{u_2} = (1,2,3) \rightarrow B(\sigma_{u_1}) = 3 + 2 + 1 = 6$$

$$\sigma_{u_3} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 3 + 3 + 2 = 8$$

$$\sigma_{u_4} = (2,2,3) \rightarrow B(\sigma_{u_1}) = 2 + 2 + 1 = 5$$

$$\sigma_{u_5} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 3 + 3 + 2 = 8$$

AGGREGATION RULE - BORDA

Compute Borda Score

$u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5$

$$\sigma_{u_1} = (1,1,1) \rightarrow B(\sigma_{u_1}) = 9$$

$$\sigma_{u_2} = (1,2,3) \rightarrow B(\sigma_{u_1}) = 6$$

$$\sigma_{u_3} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 8$$

$$\sigma_{u_4} = (2,2,3) \rightarrow B(\sigma_{u_1}) = 5$$

$$\sigma_{u_5} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 8$$

AGGREGATION RULE - BORDA

Order by Borda Score

u_1 u_2 u_3 u_4 u_5

$$\sigma_{u_1} = (1,1,1) \rightarrow B(\sigma_{u_1}) = 9$$

$$\sigma_{u_3} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 8$$

$$\sigma_{u_5} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 8$$

$$\sigma_{u_2} = (1,2,3) \rightarrow B(\sigma_{u_1}) = 6$$

$$\sigma_{u_4} = (2,2,3) \rightarrow B(\sigma_{u_1}) = 5$$

TYPE-ORDERING AGGREGATION RULE

	Borda	Type-ordering aggregation rule
Ordering rule	Borda score	Optimal
Number of possible levels	$\mathcal{O}(k^2)$	$\mathcal{O}(a^k)$

THEORETICAL ANALYSIS

ASSUMPTIONS

∞ many students

…

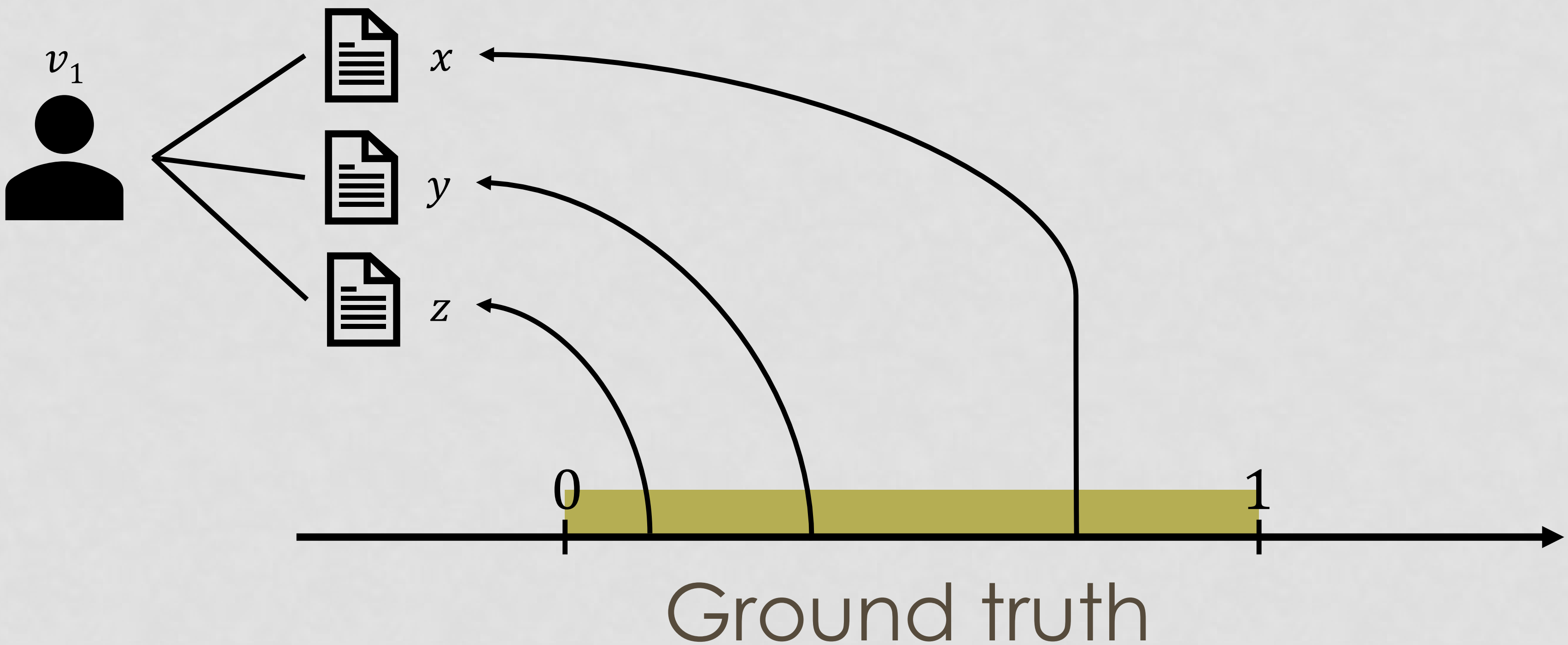
ASSUMPTIONS

Ground truth

Lower number means better student

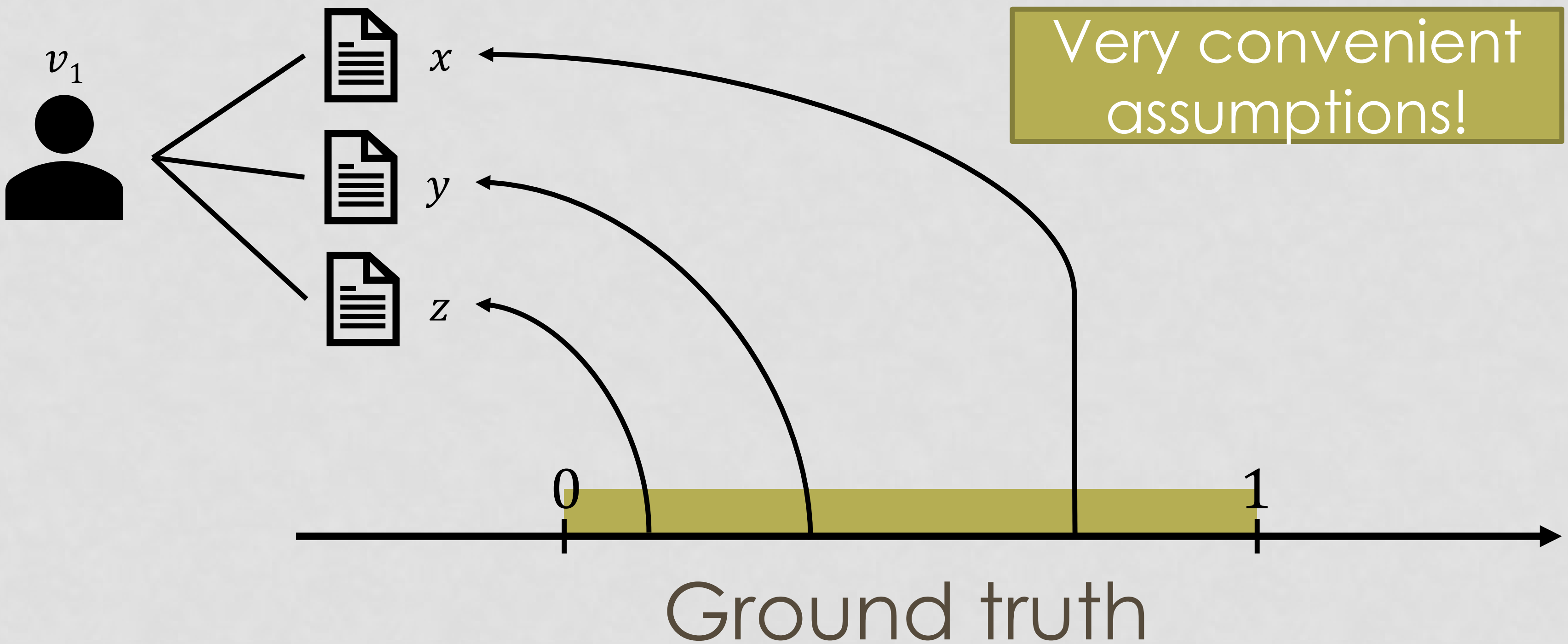
ASSUMPTIONS

Exams in a bundle ~ i. i. d.

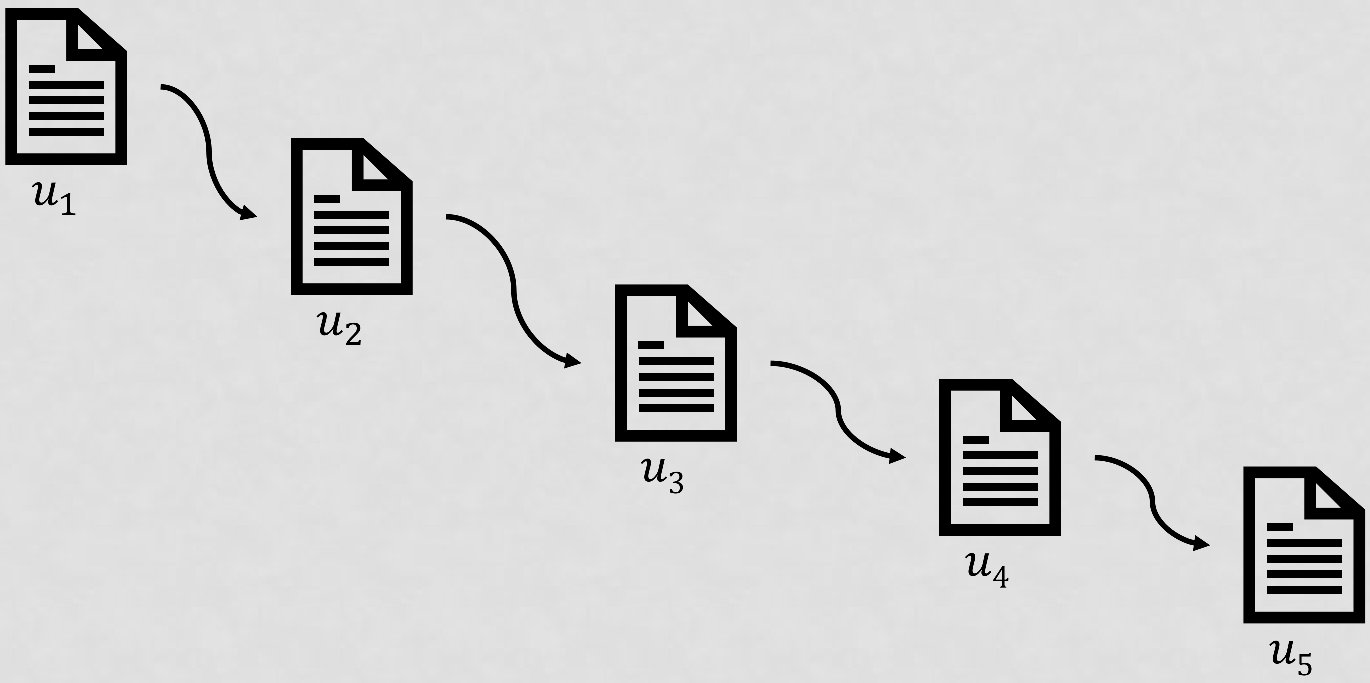


ASSUMPTIONS

Exams in a bundle ~ i. i. d.



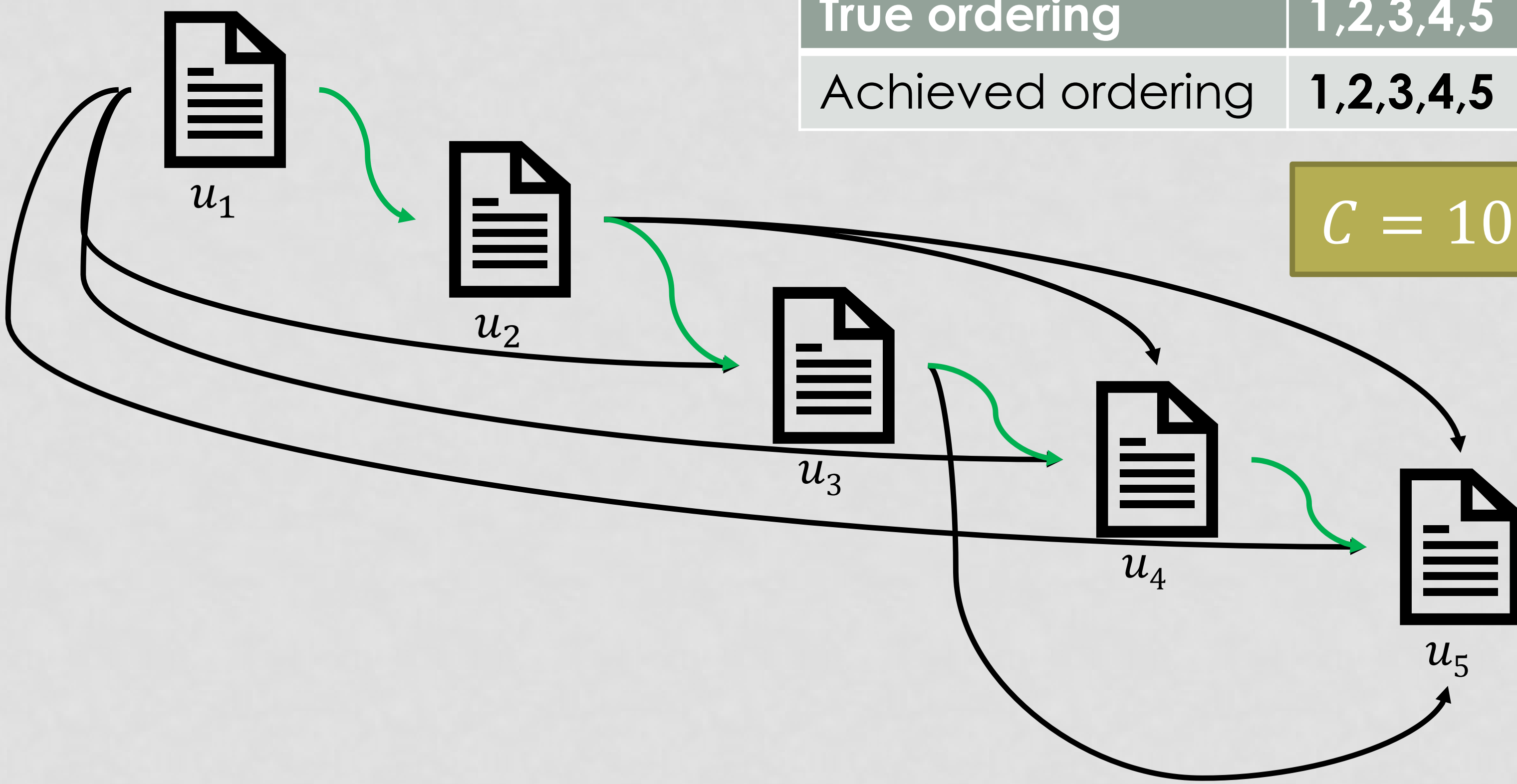
EFFICIENCY



EFFICIENCY

True ordering	1,2,3,4,5
Achieved ordering	**1,2,3,4,5**

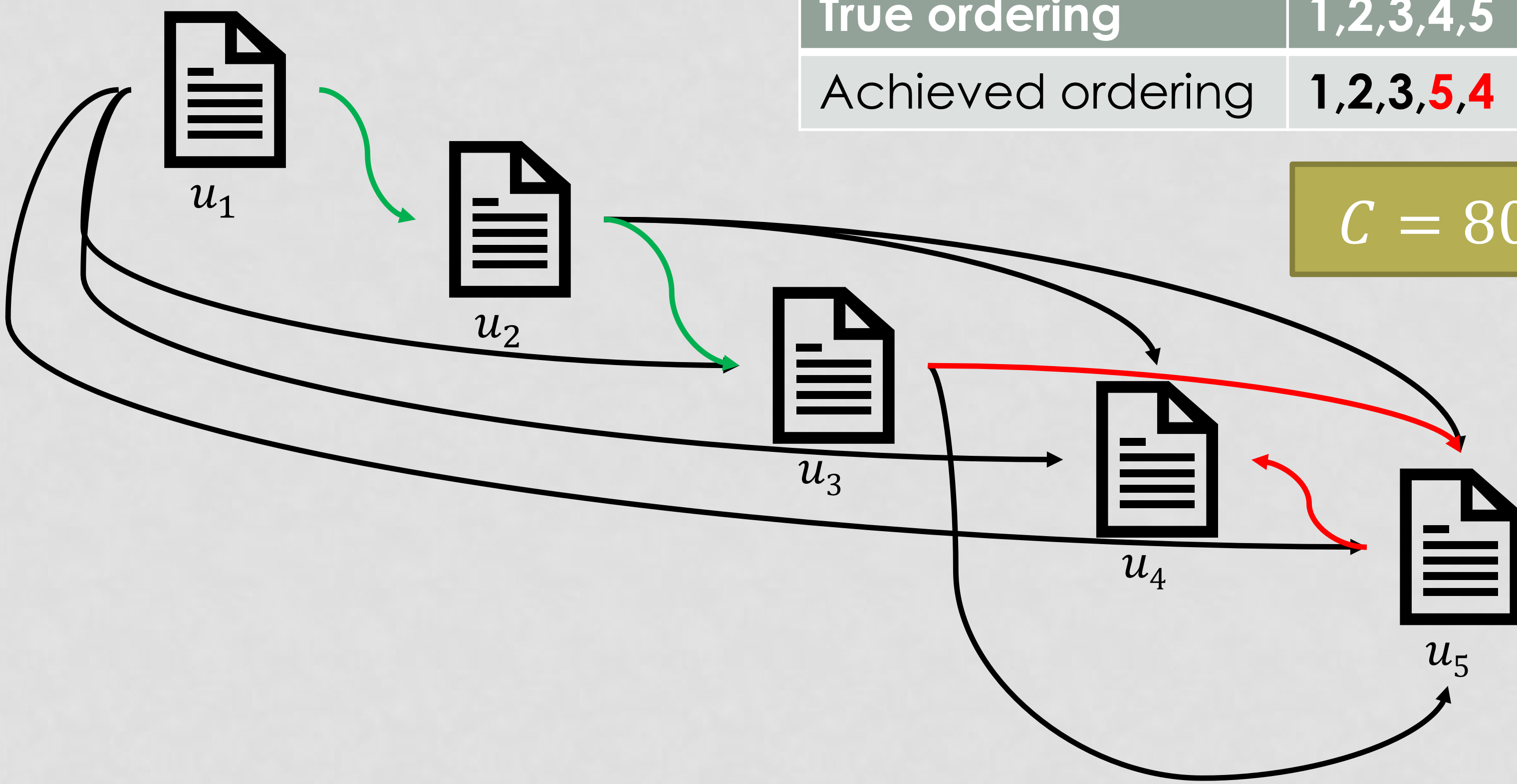
$C = 100\%$



EFFICIENCY

True ordering	1,2,3,4,5
Achieved ordering	1,2,3,5,4

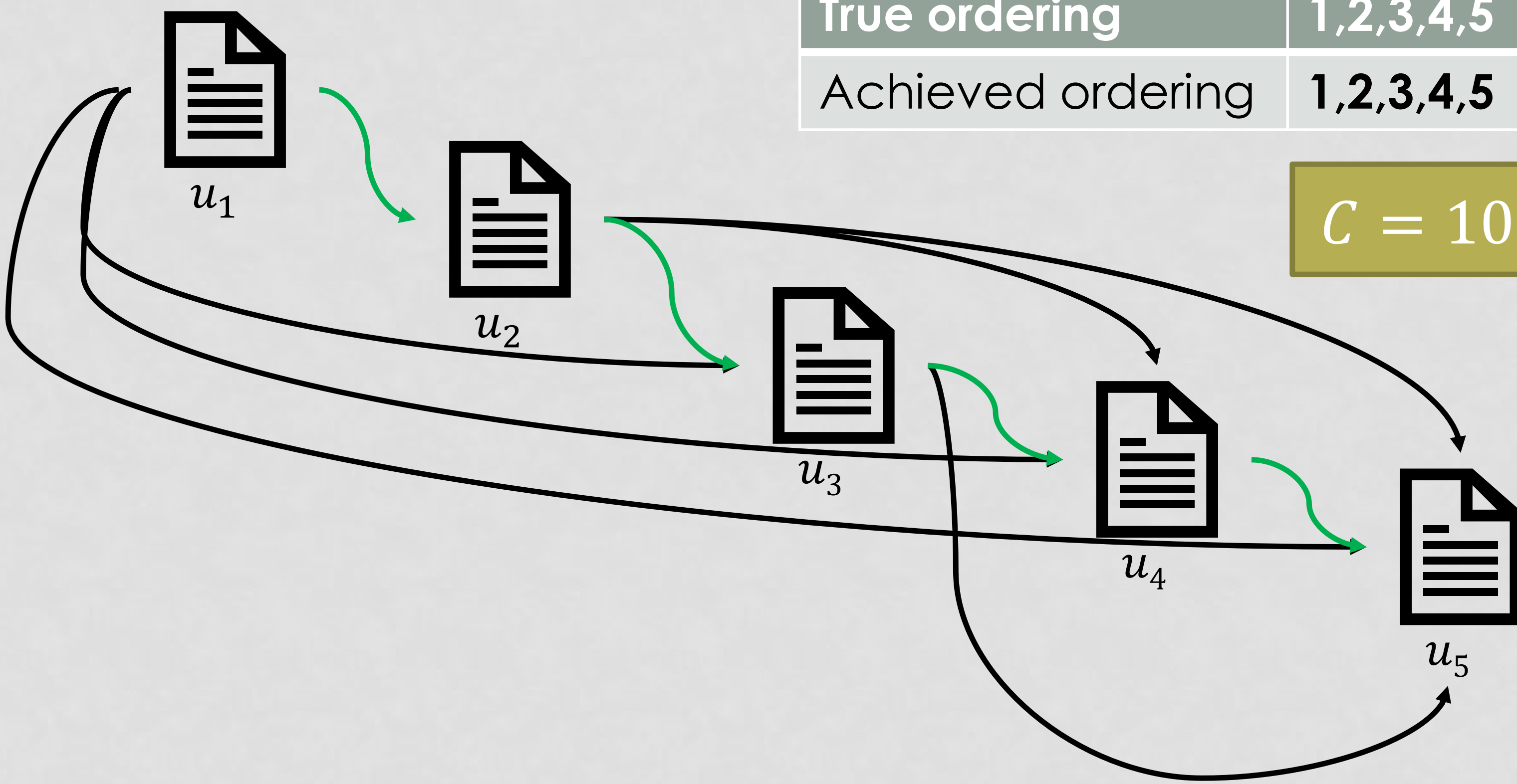
$C = 80\%$



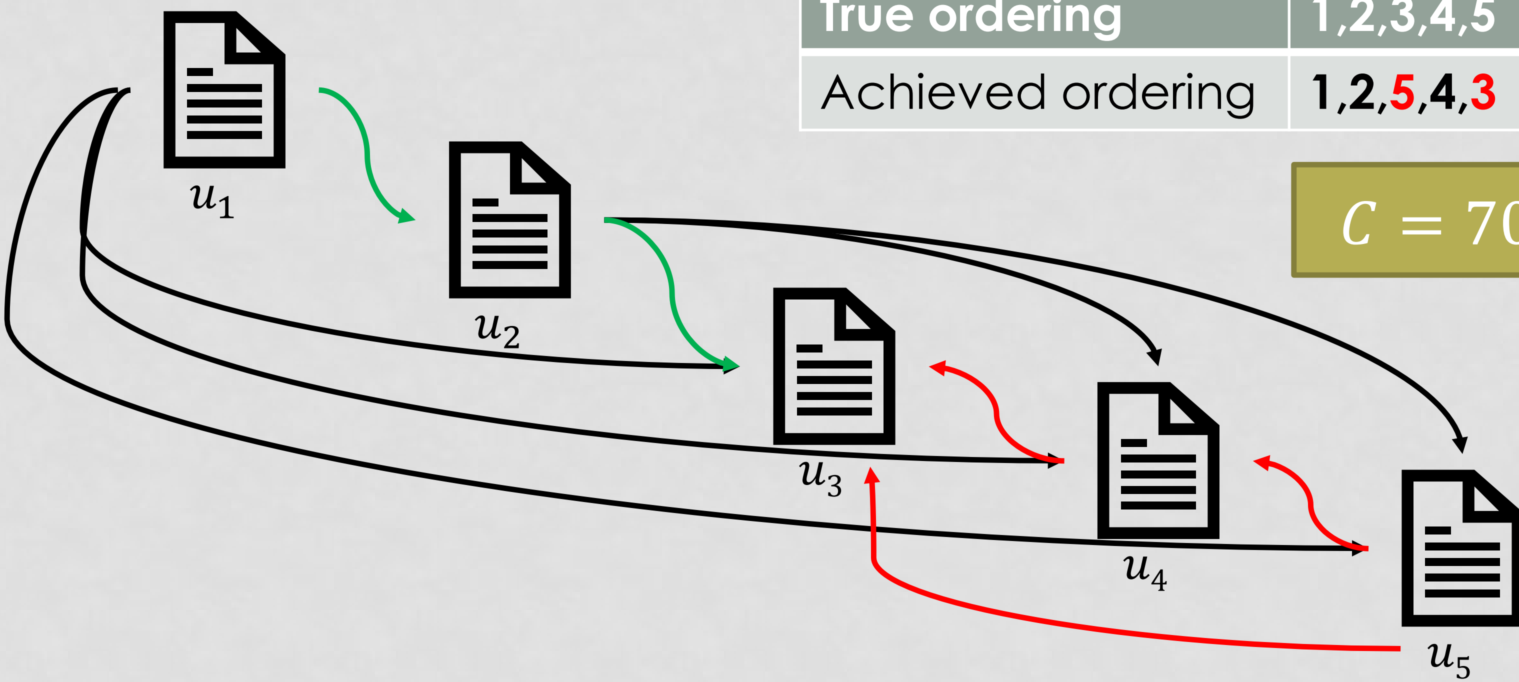
EFFICIENCY

True ordering	1,2,3,4,5
Achieved ordering	**1,2,3,4,5**

$C = 100\%$



EFFICIENCY



True ordering	1,2,3,4,5
Achieved ordering	**1,2,5,4,3**

$C = 70\%$

EXPECTED EFFICIENCY

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma] \right) dydx$$

EXPECTED FRACTION OF CORRECTLY RECOVERED PAIRWISE RELATIONS

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma] \right) dydx$$

Wanted: x has better type as y

EXPECTED FRACTION OF CORRECTLY RECOVERED PAIRWISE RELATIONS

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma] \right) dy dx$$

Less wanted: x has the same type as y

EXPECTED FRACTION OF CORRECTLY RECOVERED PAIRWISE RELATIONS

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \ and \ y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \ and \ y \rhd \sigma] \right) dydx$$

$\sigma \succ \sigma' \Rightarrow \sigma$ is a better type

EXPECTED FRACTION OF CORRECTLY RECOVERED PAIRWISE RELATIONS

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma] \right) dydx$$

For all exam papers $1 > y > x$

EXPECTED FRACTION OF CORRECTLY RECOVERED PAIRWISE RELATIONS

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma] \right) dydx$$

For all exam papers $1 > x > 0$

WEIGHTS

$$\hat{C} = \int_0^1 \int_x^1 \left(\sum_{\sigma,\sigma' : \sigma \succ \sigma'} \mathbb{P}[x \rhd \sigma \ and \ y \rhd \sigma'] + \frac{1}{2} \sum_{\sigma} \mathbb{P}[x \rhd \sigma \ and \ y \rhd \sigma] \right) dydx$$

$$\hat{C} = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma \ and \ y \rhd \sigma'] dydx \ + \frac{1}{2} \sum_{\sigma} \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma \ and \ y \rhd \sigma] dydx$$

WEIGHTS

$$\hat{C} = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma \text{ and } y \rhd \sigma'] dy dx \; + \frac{1}{2} \sum_{\sigma} \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma \text{ and } y \rhd \sigma] dy dx$$

$$C = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma,\sigma)$$

Probability Theory

weight: $W(\sigma,\sigma') = \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma \text{ and } y \rhd \sigma'] dy dx$

Calculus

Linear Algebra

RESULTS

$$\hat{C} = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma,\sigma)$$

$$W(\sigma,\sigma') = \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma] \cdot \mathbb{P}[y \rhd \sigma'] dy dx$$

Probabilities are **polynomials** → integrals can be **analytically** solved!

RESULTS

$$\hat{C} = \sum_{\sigma,\sigma':\sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2}\sum_{\sigma} W(\sigma,\sigma)$$

$$W(\sigma,\sigma') = \int_0^1 \int_x^1 \mathbb{P}[x \rhd \sigma] \cdot \mathbb{P}[y \rhd \sigma'] dy dx$$

Weights are easy to compute (closed form solution)

CONCLUSION

$$\hat{C}(k, \succ, P) = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma,\sigma)$$

OPTIMIZATION

$$\hat{C}(k, \succ, P) = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma, \sigma)$$

$$\max_{\succ} \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma, \sigma)$$

Weights are independent of $\succ$ → computed only once

ADDING ELASTICITY

$$W(\sigma,\sigma') = \int_0^1 \int_x^1 \boldsymbol{f(x,y)}\,\mathbb{P}[x \rhd \sigma] \cdot \mathbb{P}[y \rhd \sigma']\,dy\,dx$$

$$\boldsymbol{f(x,y)} = \begin{cases} 1 \; if \; y - x \geq 5\% \\ 0 \; otherwise \end{cases}$$

ADDING ELASTICITY

$$W(\sigma,\sigma') = \int_0^1 \int_x^1 \boldsymbol{f(x,y)} \mathbb{P}[x \rhd \sigma] \cdot \mathbb{P}[y \rhd \sigma'] dy dx$$

$$\boldsymbol{f(x,y)} = \begin{cases} 1 \; if \; x \leq 20\% \\ 0 \; otherwise \end{cases}$$

OPTIMIZATION

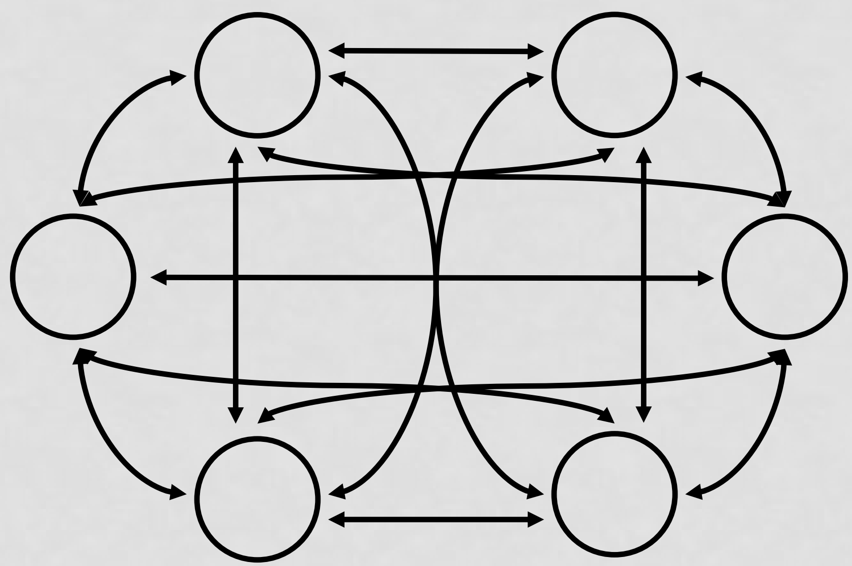
$$\hat{C}(k, \succ, P, \boldsymbol{f}) = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma, \sigma)$$

$$\max_{\succ} \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma, \sigma)$$

Weights are **still** independent of $\succ$ → computed only once

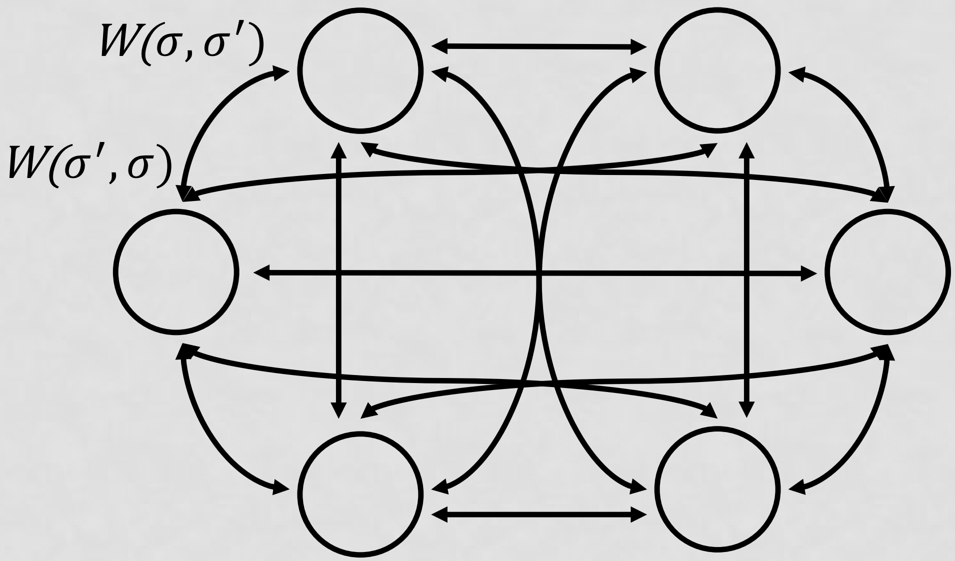
REFORMULATE PROBLEM

$$\max_{\succ} \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma,\sigma)$$



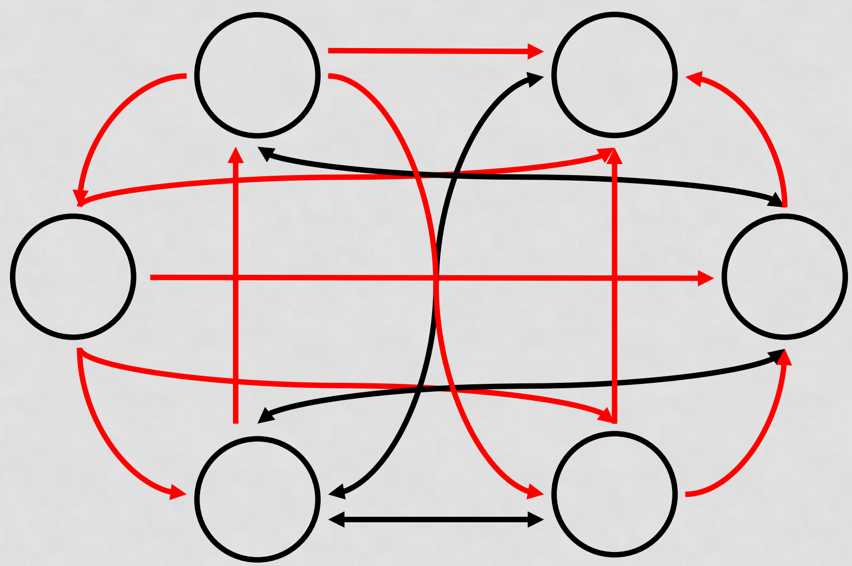
REFORMULATE PROBLEM

$$\max_{\succ} \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2}\sum_{\sigma} W(\sigma,\sigma)$$



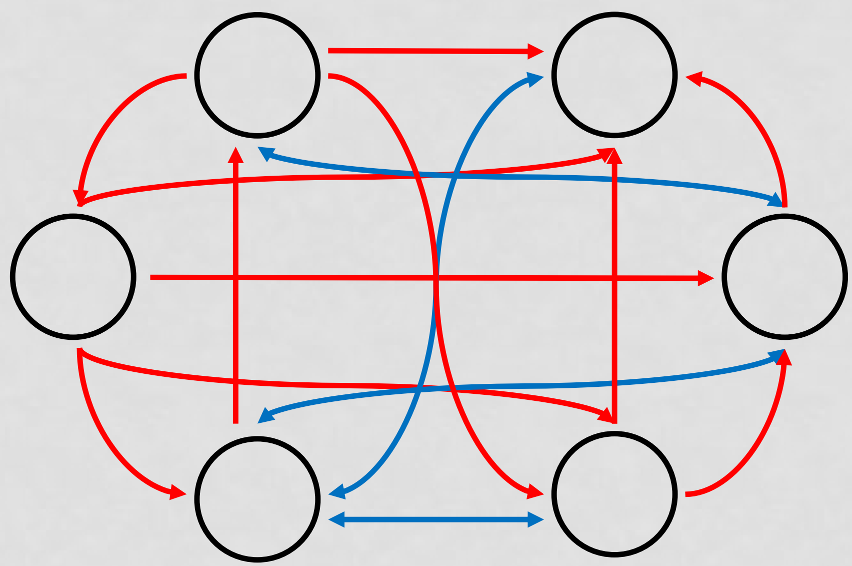
AUXILIARY GRAPH

Different weights ⇒ Keep edge with bigger weight



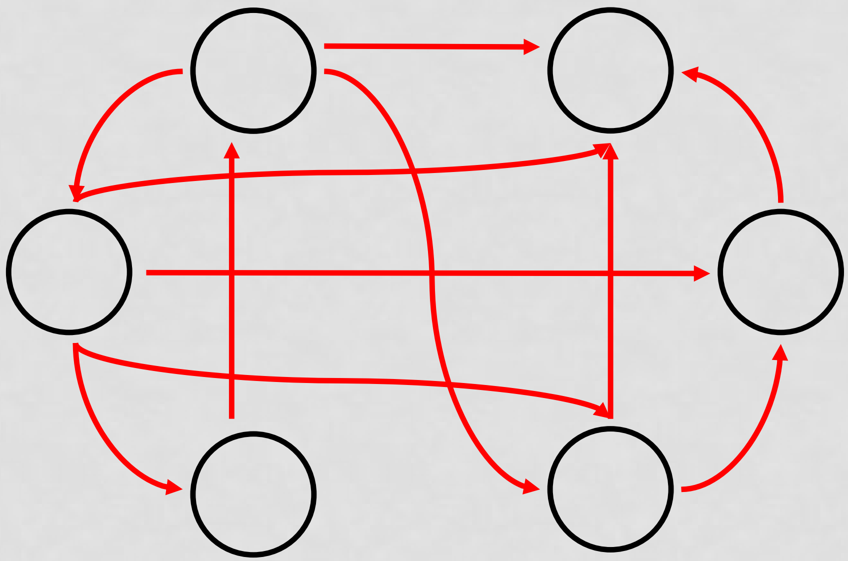
AUXILIARY GRAPH

Equal weights ⇒ discard



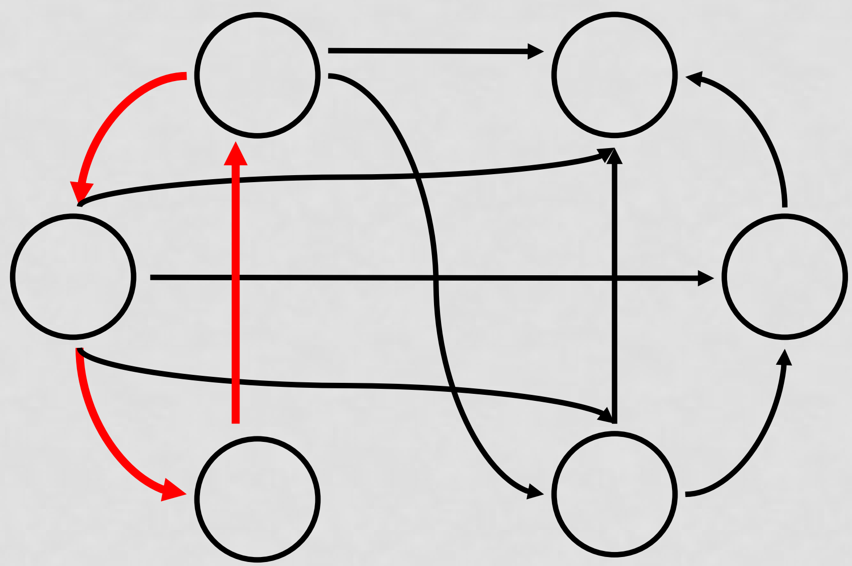
AUXILIARY GRAPH

Equal weights ⇒ discard



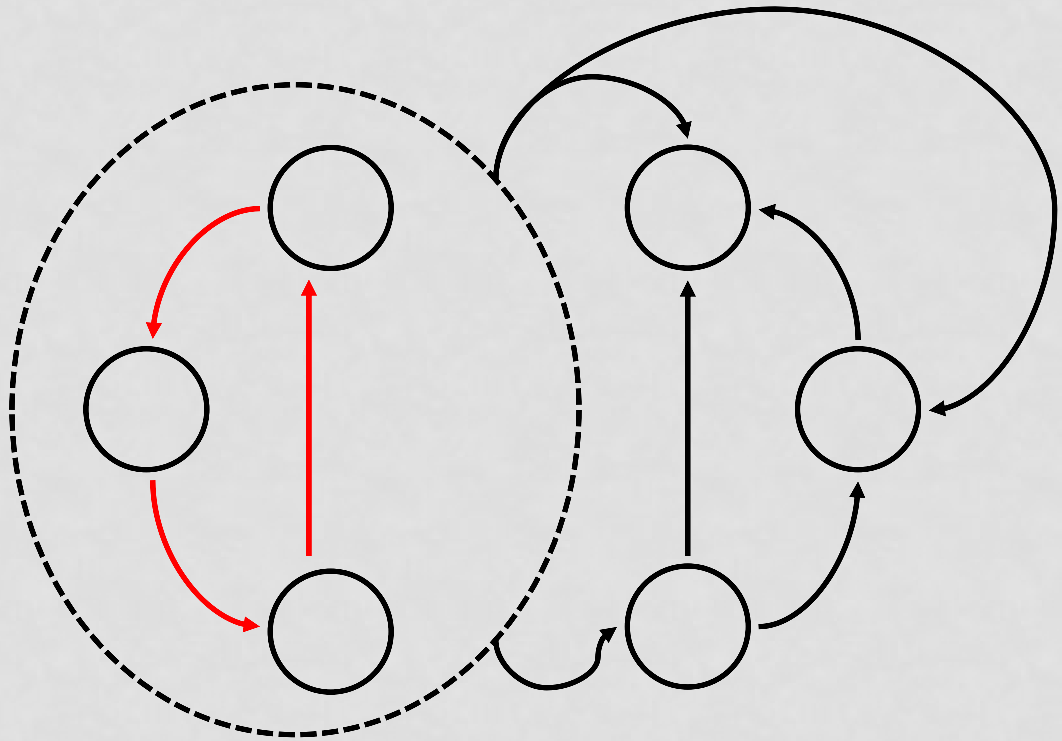
AUXILIARY GRAPH

Strongly connected regions (cycles)



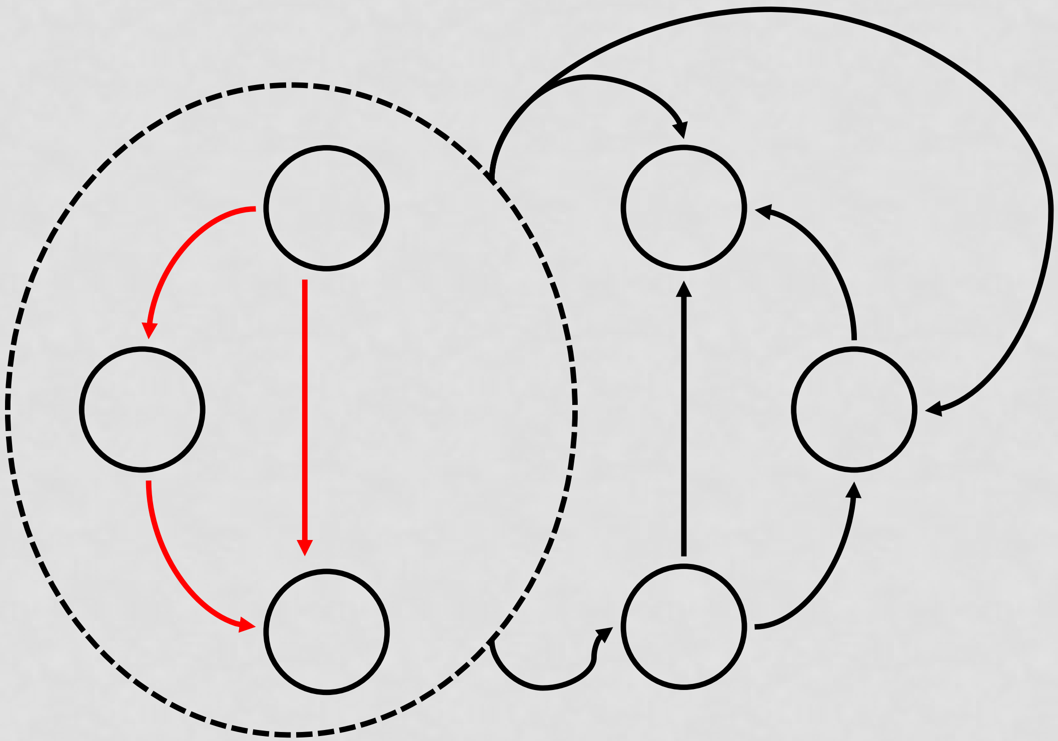
AUXILIARY GRAPH

Strongly connected regions (cycles)



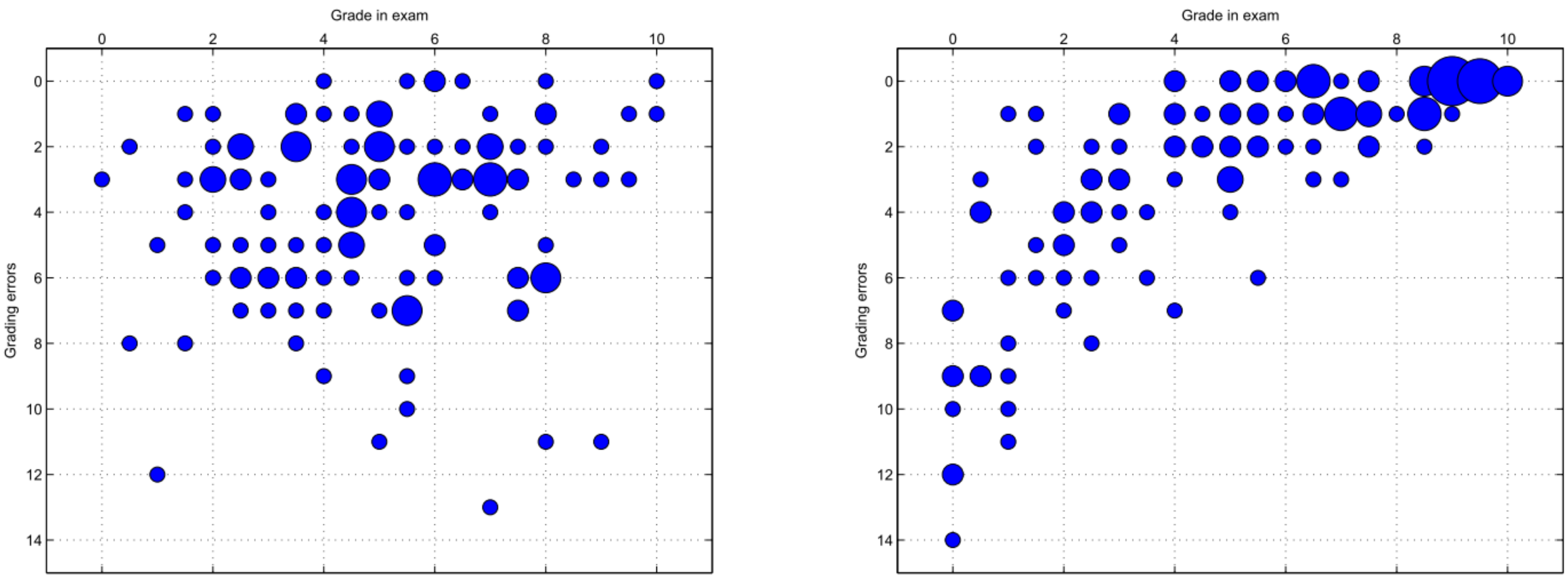
AUXILIARY GRAPH

Brute force (or Borda in case of large cycle)



FIELD EXPERIMENT

$$k = 6, n = 136$$



(a) Realistic noise (b) Mallows noise

FIELD EXPERIMENT

$$P_{\text{real}} = \begin{bmatrix} 0.463 & 0.257 & 0.102 & 0.058 & 0.058 & 0.058 \\ 0.205 & 0.316 & 0.227 & 0.110 & 0.066 & 0.073 \\ 0.161 & 0.191 & 0.257 & 0.205 & 0.132 & 0.051 \\ 0.102 & 0.117 & 0.191 & 0.242 & 0.279 & 0.066 \\ 0.044 & 0.066 & 0.139 & 0.220 & 0.301 & 0.227 \\ 0.022 & 0.051 & 0.080 & 0.161 & 0.161 & 0.522 \end{bmatrix}$$

$$P_{\text{mallows}} = \begin{bmatrix} 0.6337 & 0.1753 & 0.0824 & 0.0494 & 0.0339 & 0.0253 \\ 0.1753 & 0.5112 & 0.1549 & 0.0768 & 0.0479 & 0.0339 \\ 0.0824 & 0.1549 & 0.4865 & 0.1500 & 0.0768 & 0.0494 \\ 0.0494 & 0.0768 & 0.1500 & 0.4865 & 0.1549 & 0.0824 \\ 0.0339 & 0.0479 & 0.0768 & 0.1549 & 0.5112 & 0.1753 \\ 0.0253 & 0.0339 & 0.0494 & 0.0824 & 0.1753 & 0.6337 \end{bmatrix}$$

SIMULATIONS

All2all:

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = 1$$

SIMULATIONS

Th-10% and Th-50%:

$$\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \begin{cases} 1 \; if \;\; x \leq th\% \\ 0 \; otherwise \end{cases}$$

SIMULATIONS

Acc-2% and Acc-5%:

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} 1 \; if \;\; y - x \geq acc\% \\ 0 \; otherwise \end{cases}$$

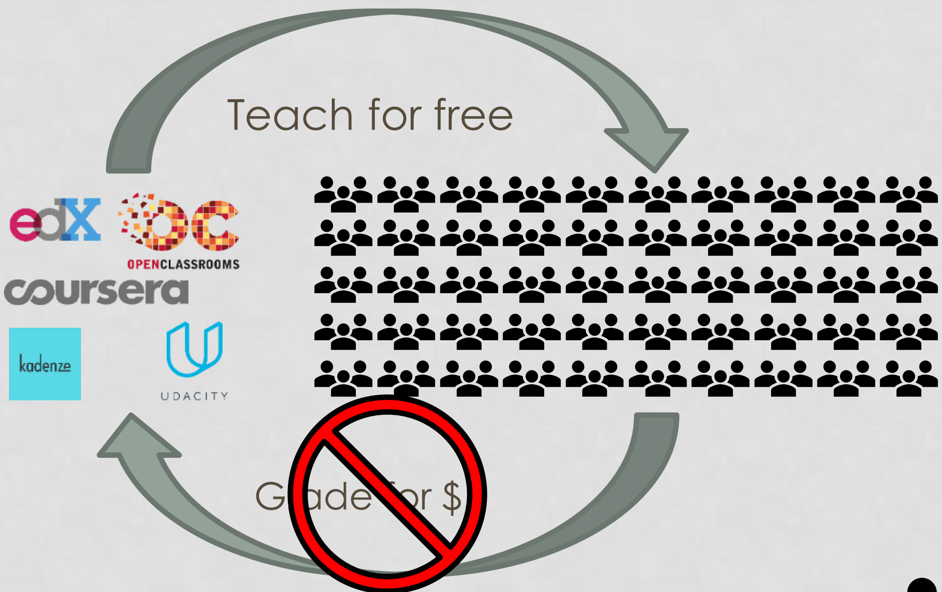
SMALL STRONGLY CONNECTED COMPONENTS

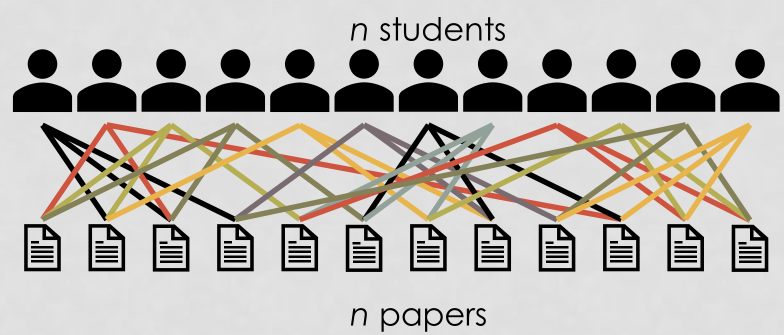
	realistic model				mallows model			
size	all2all	th-50%	acc-2%	acc-5%	all2all	th-50%	acc-2%	acc-5%
1	448	460	449	451	453	459	449	449
3–7	13	2	12	10	6	3	10	12
8–11	1	0	1	1	2	0	2	0
≥ 12	0	0	0	0	1	0	1	1
max	10	3	10	10	20	4	20	20

PERFORMANCE

noise	perfect grading		realistic grading				mallows grading			
setting	theory	$n = 10^4$	theory		$n = 10^4$		theory		$n = 10^4$	
method	borda	borda	opt	borda	opt	borda	opt	borda	opt	borda
all2all	92.01	92.02	80.01	79.57	80.09	79.57	85.15	84.38	85.16	84.39
th-10%	96.94	96.95	87.61	87.18	87.60	87.17	92.05	90.52	92.07	90.54
th-50%	94.13	94.14	83.62	83.43	83.62	83.43	88.39	87.80	88.40	87.81
acc-2%	93.57	93.57	81.27	80.73	81.27	80.74	86.52	85.72	86.52	85.73
acc-5%	95.47	95.47	82.97	82.42	82.97	82.42	88.42	87.61	88.42	87.62

CONCLUSION


Teach for free
OPENCLASSROOMS
coursera
kadenze
UDACITY
Grade for $


n students
n papers

CONCLUSION

$$v_1 = \begin{bmatrix} \mathbf{u_3} \\ u_4 \\ u_2 \end{bmatrix} \quad v_2 = \begin{bmatrix} \mathbf{u_3} \\ u_5 \\ u_1 \end{bmatrix} \quad v_4 = \begin{bmatrix} u_7 \\ \mathbf{u_3} \\ u_1 \end{bmatrix}$$

Type – grading result of exam paper

$$\sigma_{u_3} = (1,1,2)$$

u_1 u_2 u_3 u_4 u_5

$$\sigma_{u_1} = (1,1,1) \rightarrow B(\sigma_{u_1}) = 9$$

$$\sigma_{u_3} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 8$$

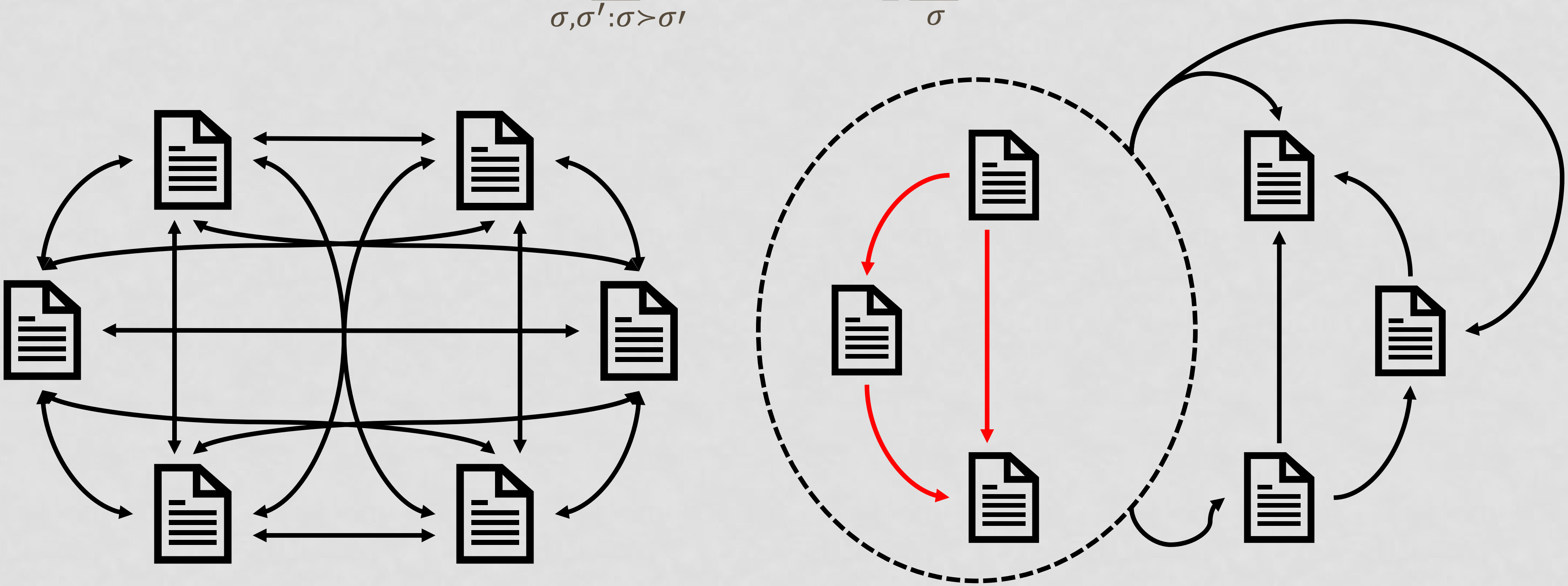
$$\sigma_{u_5} = (1,1,2) \rightarrow B(\sigma_{u_1}) = 8$$

$$\sigma_{u_2} = (1,2,3) \rightarrow B(\sigma_{u_1}) = 6$$

$$\sigma_{u_4} = (2,2,3) \rightarrow B(\sigma_{u_1}) = 5$$

CONCLUSION

$$\max_{\succ} \sum_{\sigma,\sigma':\sigma\succ\sigma'} W(\sigma,\sigma') + \frac{1}{2}\sum_{\sigma} W(\sigma,\sigma)$$



CONCLUSION

noise	perfect grading		realistic grading				mallows grading			
setting	theory	$n = 10^4$	theory		$n = 10^4$		theory		$n = 10^4$	
method	borda	borda	opt	borda	opt	borda	opt	borda	opt	borda
all2all	92.01	92.02	80.01	79.57	80.09	79.57	85.15	84.38	85.16	84.39
th-10%	96.94	96.95	87.61	87.18	87.60	87.17	92.05	90.52	92.07	90.54
th-50%	94.13	94.14	83.62	83.43	83.62	83.43	88.39	87.80	88.40	87.81
acc-2%	93.57	93.57	81.27	80.73	81.27	80.74	86.52	85.72	86.52	85.73
acc-5%	95.47	95.47	82.97	82.42	82.97	82.42	88.42	87.61	88.42	87.62

What about combining filters?

How to interpret a grade?

THANK YOU!

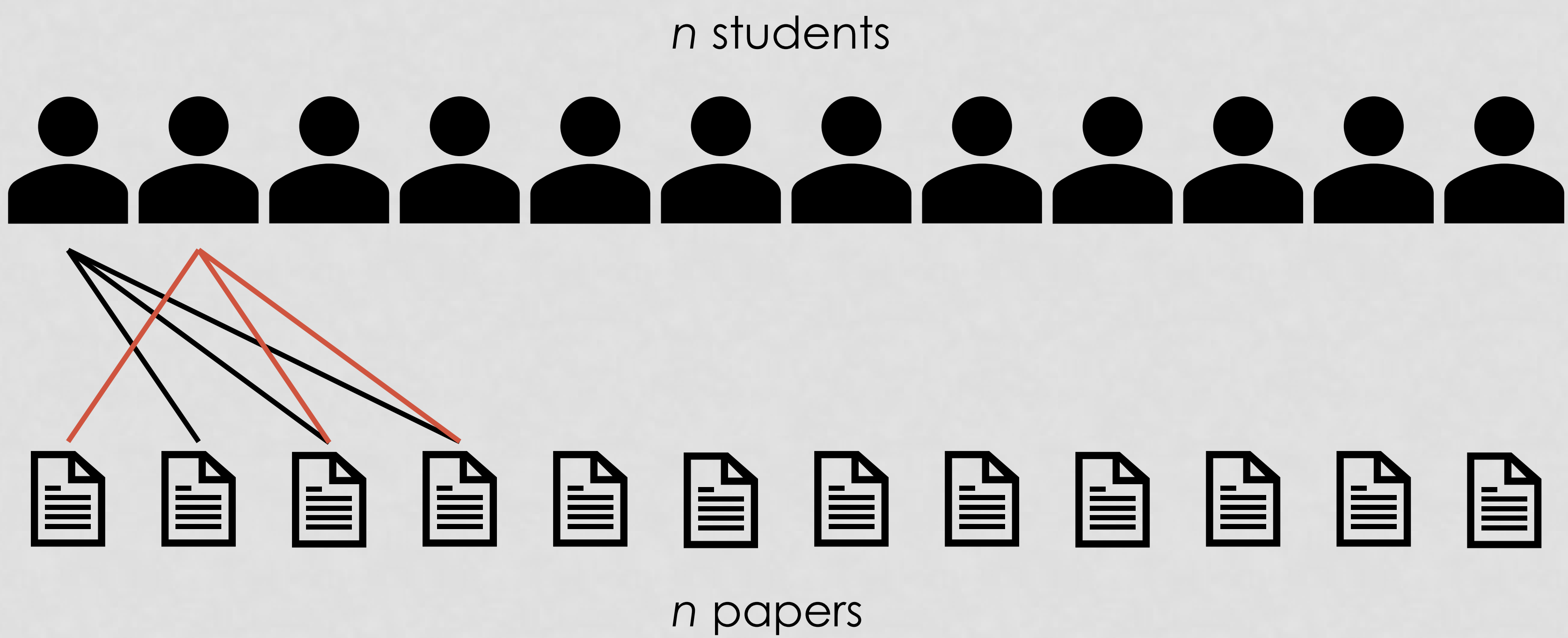
PLEASE ASK QUESTIONS

ADDITIONAL SLIDES

TABLE OF CONTENT

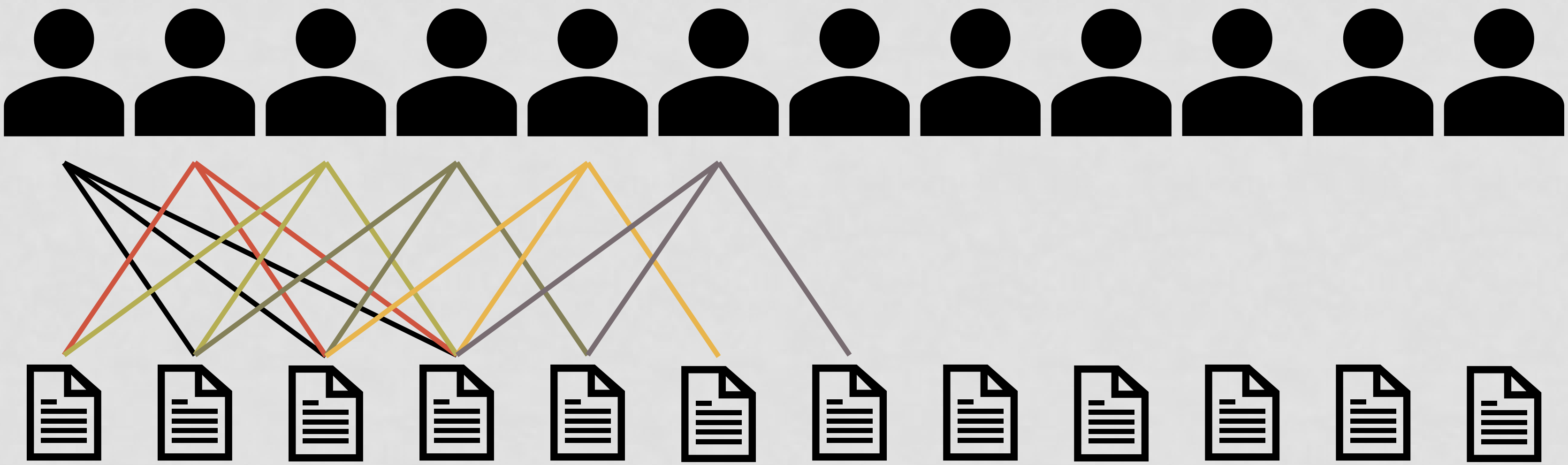
WRONG BUNDLING EXAMPLE

SETTING



SETTING

n students



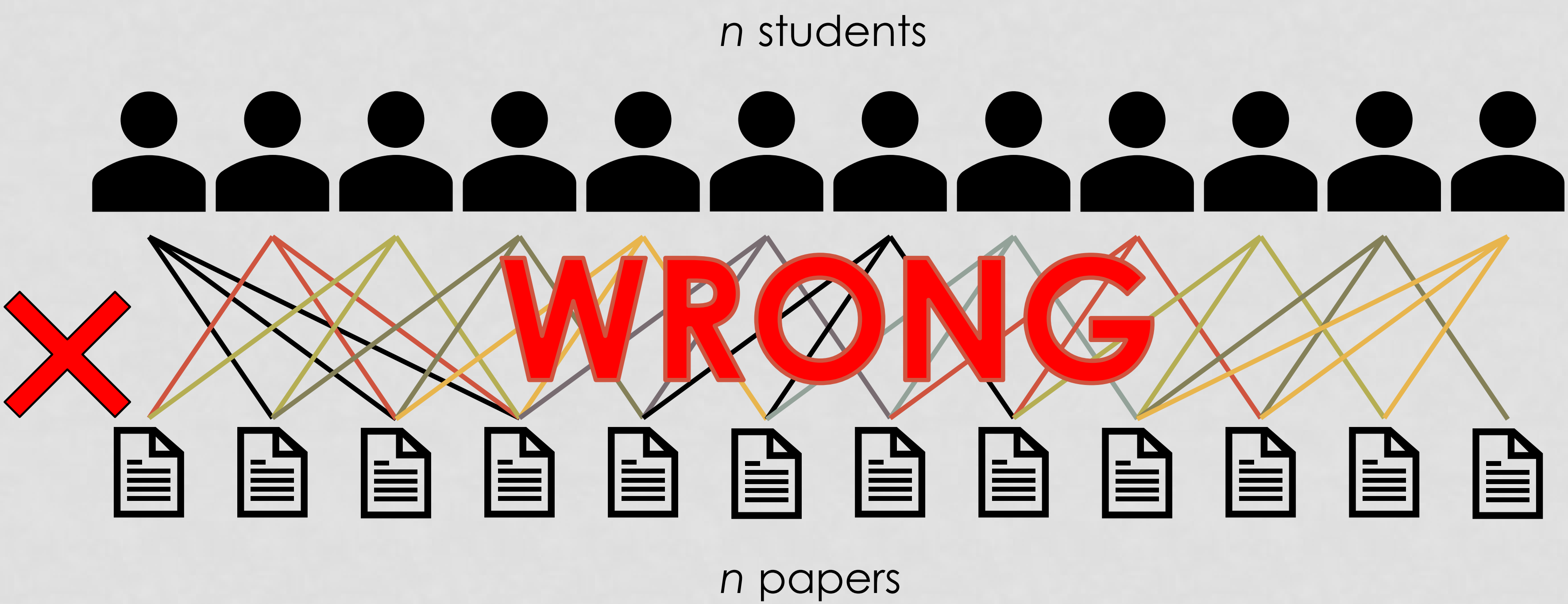
n papers

SETTING

n students

n papers

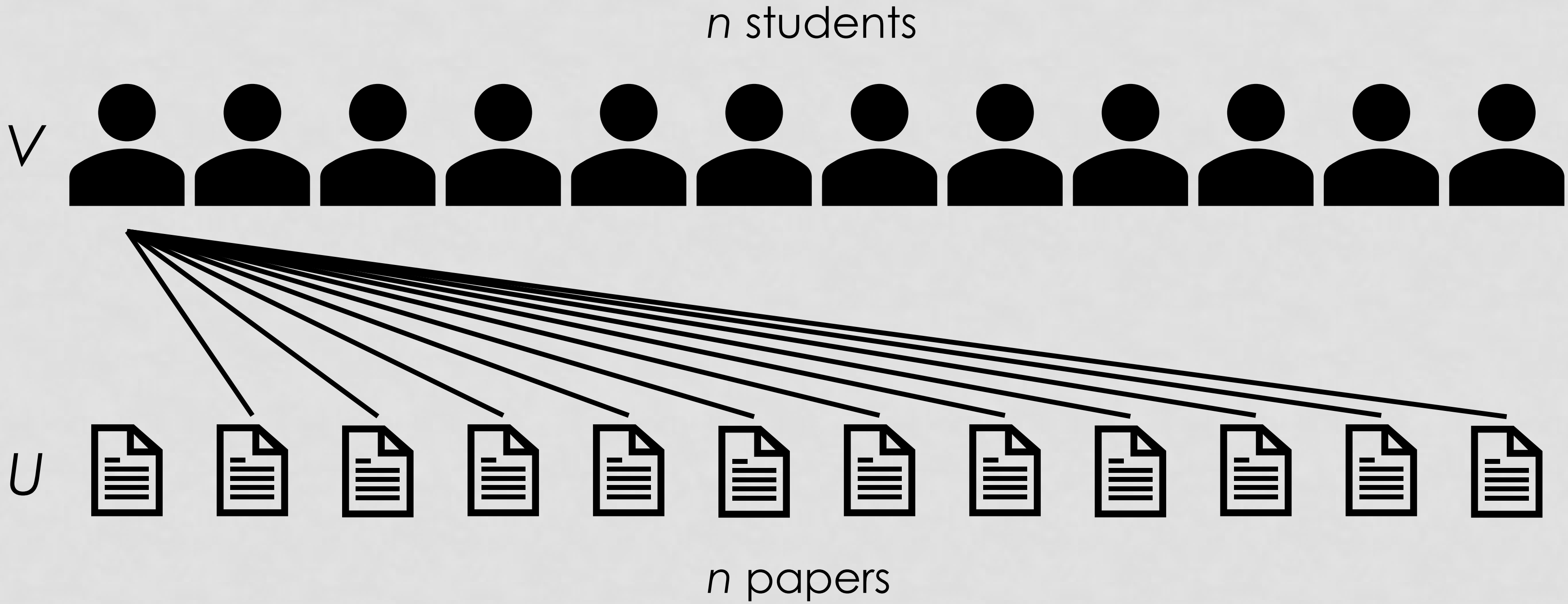
SETTING



BUILDING BUNDLE GRAPH

HOW TO BUILD BUNDLE GRAPH

- *Start from complete bipartite graph $K_{n,n}$ (all graders connected to all papers),*
- *Remove the edges between graders and their own papers,*
- *Draw a perfect matching uniformly at random among all perfect matchings (that do not include previously removed edges),*
- *Repeat previous step until each grader has k papers (and each paper has 3 graders)*



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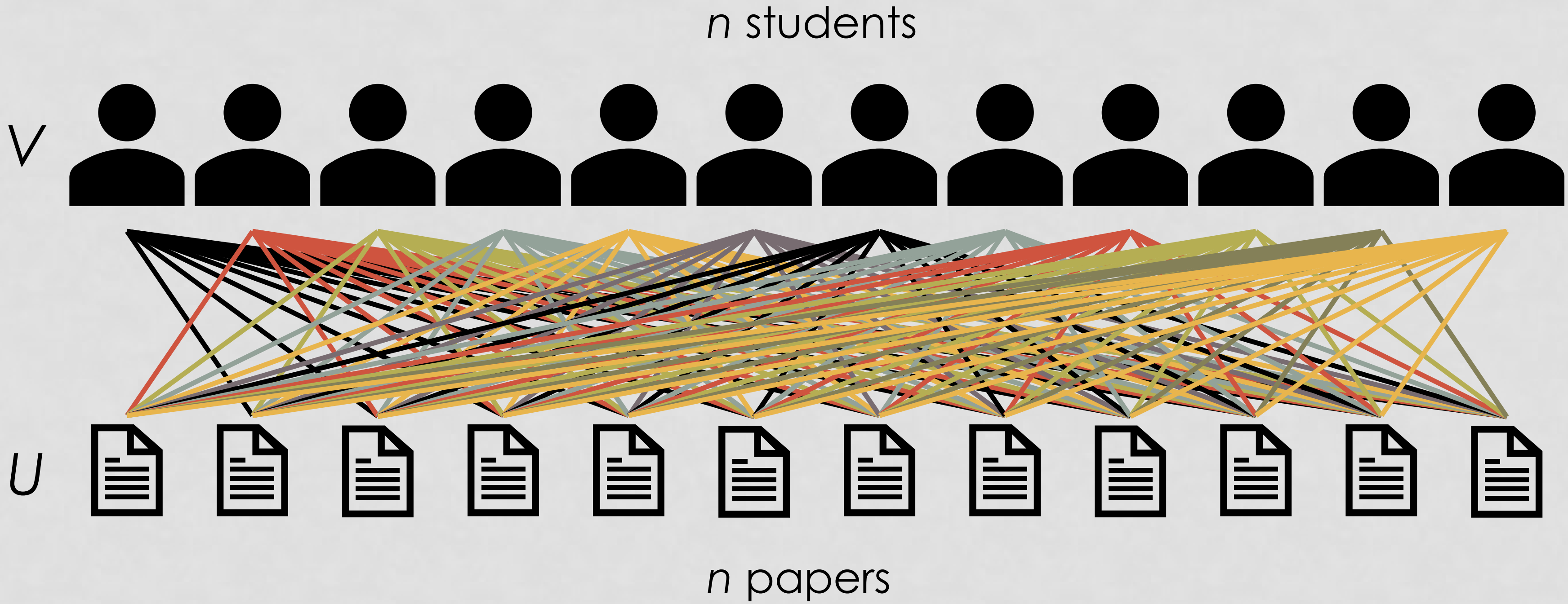

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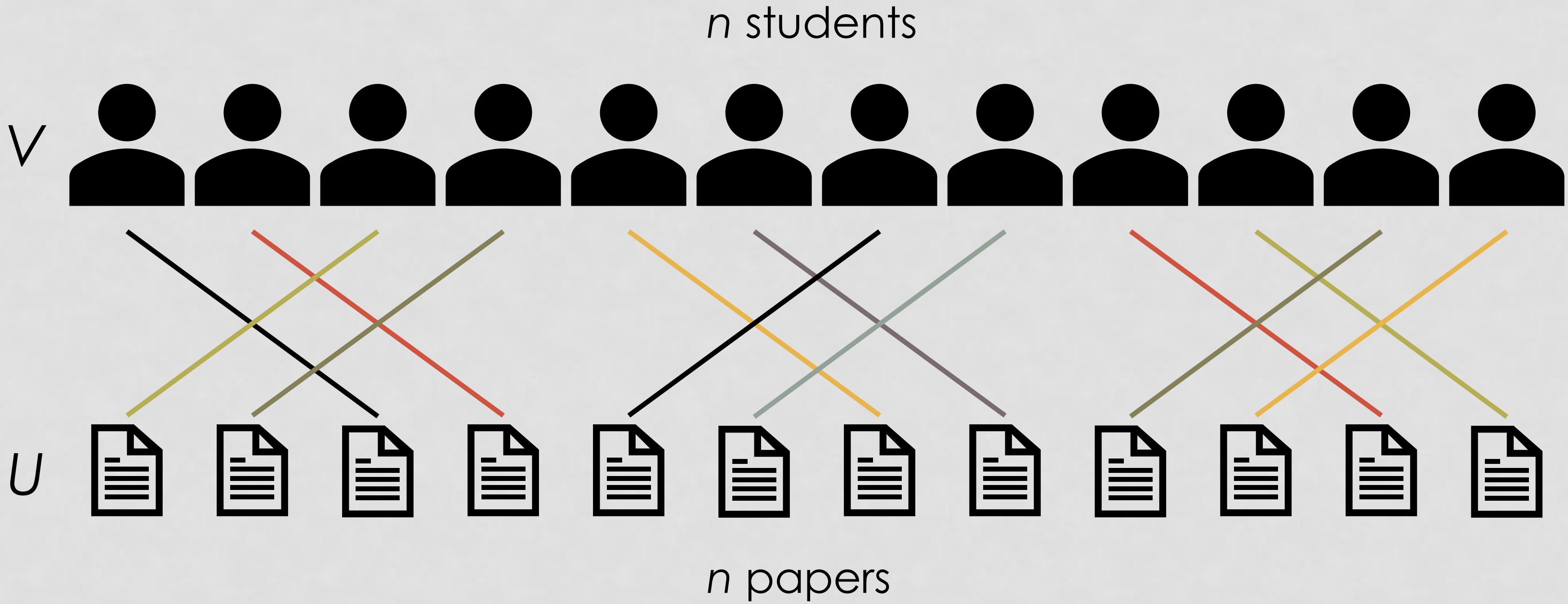

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- *Start from complete bipartite graph $K_{n,n}$ (all graders connected to all papers),*
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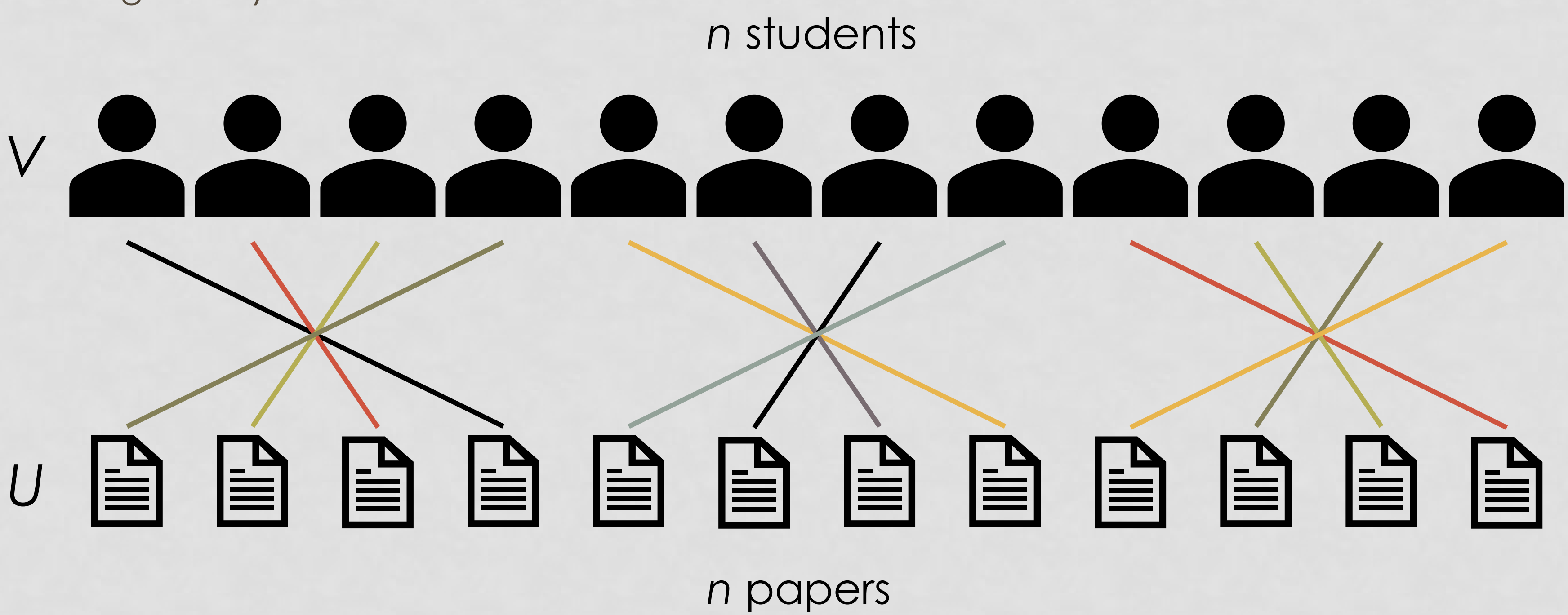

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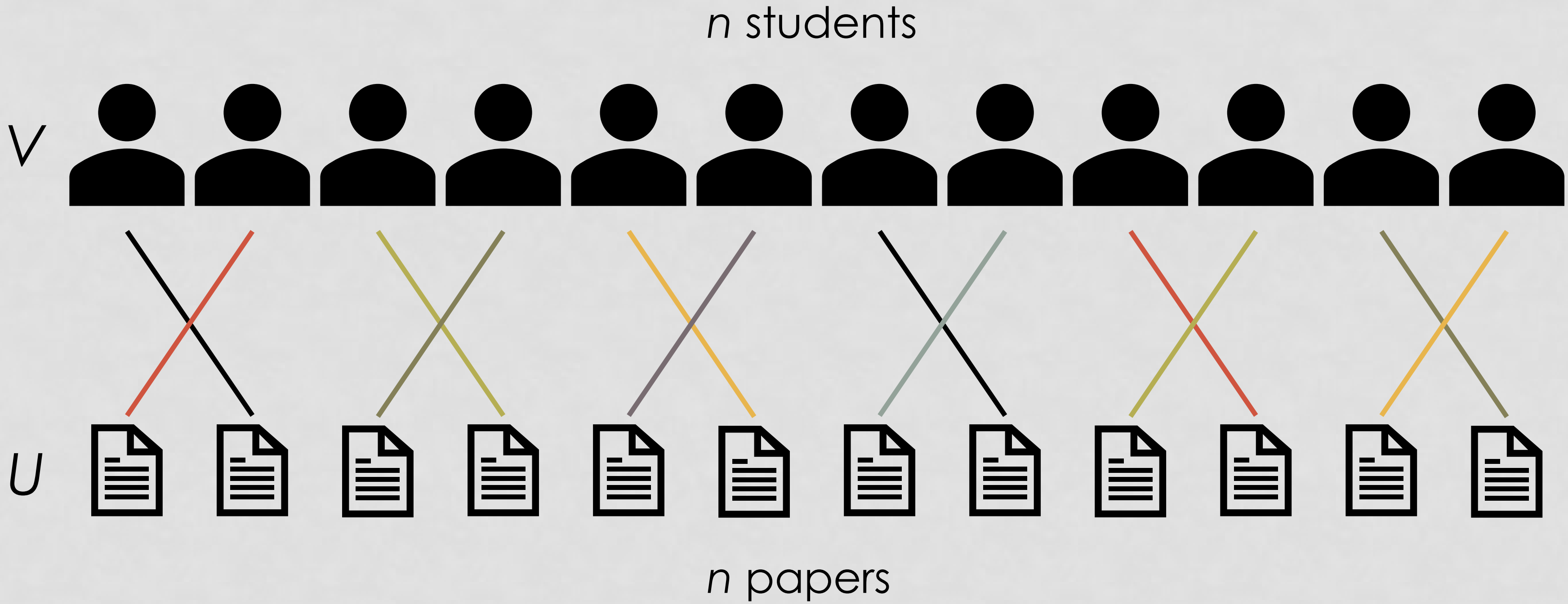

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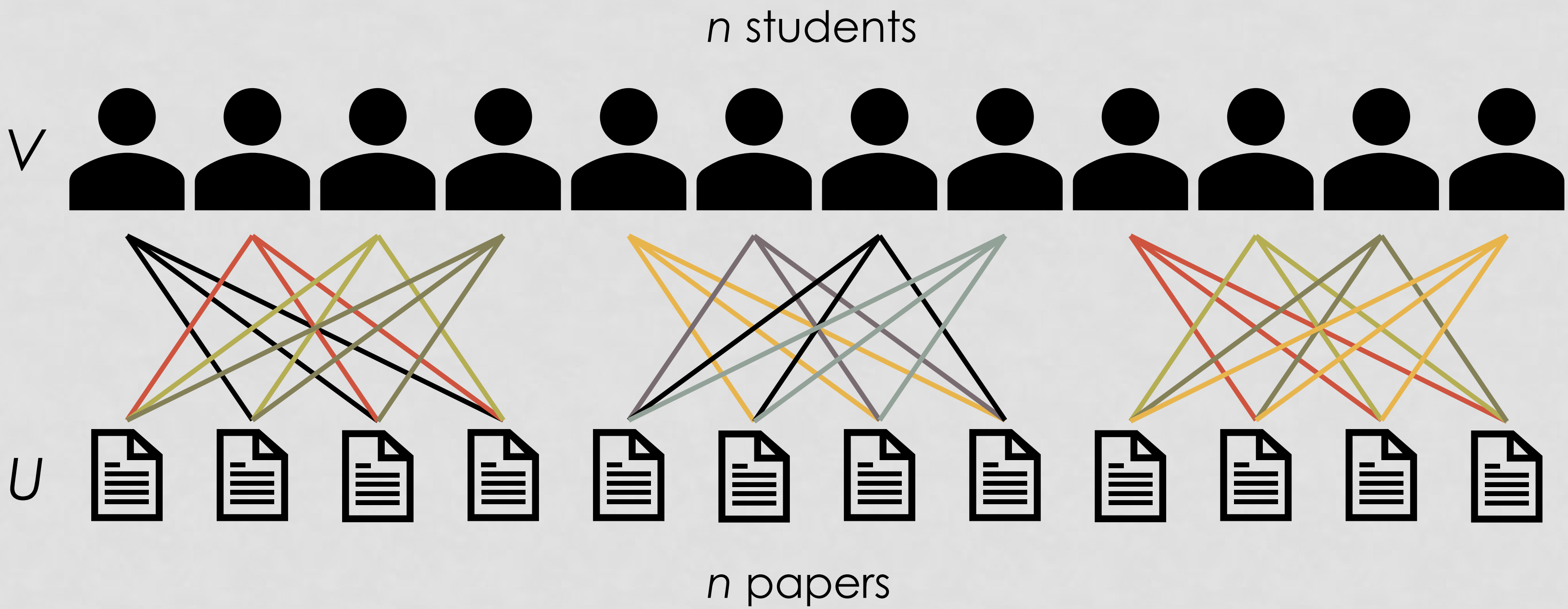
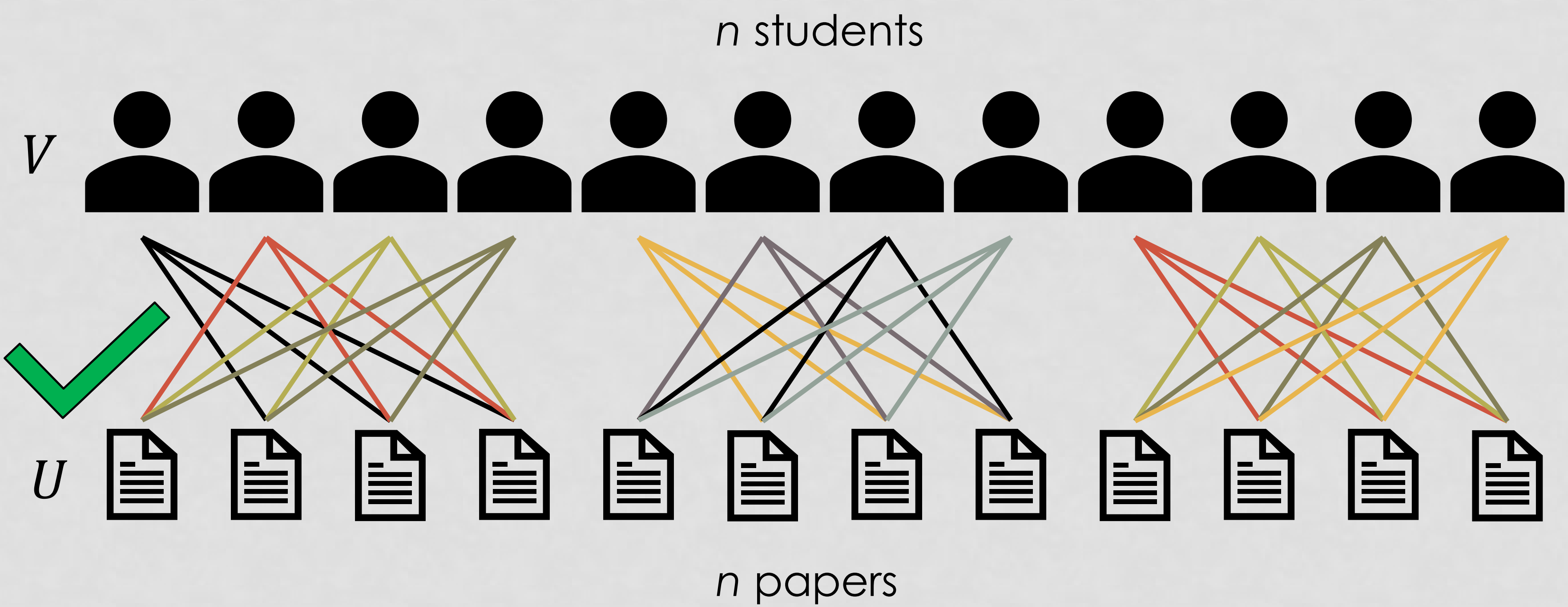


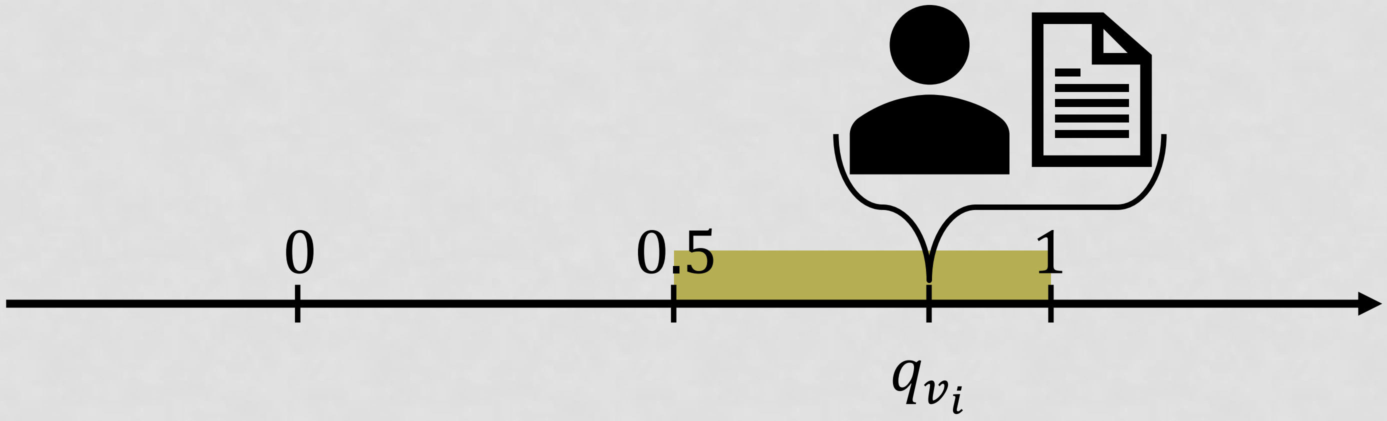
SETTING

- k-regular bipartite graph $G = (U, V, E)$ with n nodes on each side. U contains exam papers and V contains graders,
- k edges from each grader v_i to k exam papers from U.
- Student cannot grade her own paper edge from v_i to u_i is forbidden for all values of i.



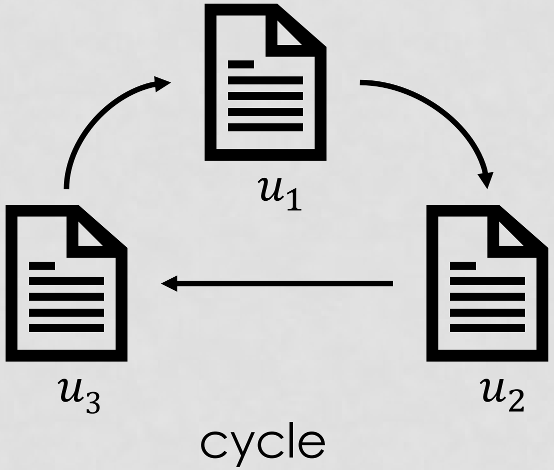
MODELLING STUDENT'S GRADING BEHAVIOUR

QUALITY $q \in \left[\frac{1}{2}, 1\right]$



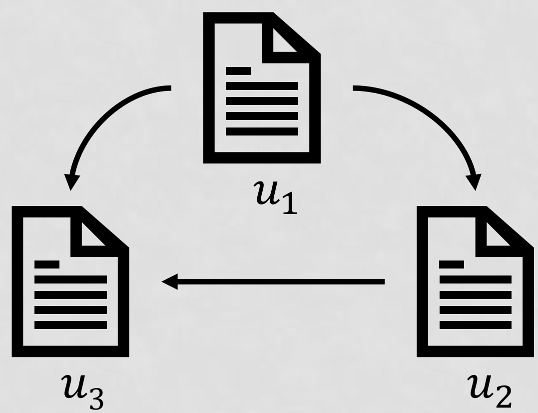
GRADING BEHAVIOUR

- Let $k = 3$, $q = 0.7$, true rank $v_i^{true} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.
- First attempt:



GRADING BEHAVIOUR

- Let $k = 3$, $q = 0.7$, true rank $v_i^{true} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$.
- Second attempt:



no cycle – grading outcome

DERIVATIONS

EXPECTED FRACTION OF CORRECTLY RECOVERED PAIRWISE RELATIONS

$$C = \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2} \sum_{\sigma} W(\sigma,\sigma)$$

From assumptions:
infinite number of students ⇒ no dependency between the rank vectors that the exam papers x and y get after grading:

$$\mathbb{P}[x \rhd \sigma \; and \; y \rhd \sigma'] = \mathbb{P}[x \rhd \sigma] \cdot \mathbb{P}[y \rhd \sigma']$$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, which is a probability that an exam paper x has type $\sigma = (\sigma_1, \ldots, \sigma_k)$.

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \dots, \sigma_k)$.

Procedure:

- Denote by $\mathcal{E}(x, \sigma_i)$ an event that i-th element of exam paper's x type is σ_i, which is equivalent to the event that the exam paper x was ranked σ_i-th in a bundle,
- Based on above determine the probability that $\sigma = (\sigma_1, \dots, \sigma_k)$ is of a particular type,
- Multiply above calculated probability by the number of ways in which such a type could have been achieved.

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \dots, \sigma_k)$.

Procedure:

- **Denote by $\mathcal{E}(\mathrm{x}, \sigma_\mathrm{i})$ an event that i-th element of exam paper's x type is σ_i, which is equivalent to the event that the exam paper x was ranked σ_i-th in a bundle,**
- **Based on above determine the probability that $\sigma = (\sigma_1, \dots, \sigma_k)$ is of a particular type,**
- Multiply above calculated probability by the number of ways in which such a type could have been achieved.

$$\sigma = (\sigma_1, \dots, \sigma_k)$$

$$\mathbb{P}[\mathcal{E}(x, \sigma_1)\ and\ \dots\ and\ \mathcal{E}(x, \sigma_k)] = \prod_{i=1}^{k} \mathbb{P}[\mathcal{E}(x, \sigma_i)]$$

Infinitely many students => the quality of each exam paper in the bundle does not affect quality of other exam papers

FINDING $\mathbb{P}[x \rhd \sigma]$

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- **Multiply above calculated probability by the number of ways in which such a type could have been achieved.**

$$\sigma = (\sigma_1, \dots, \sigma_k)$$

$$\mathbb{P}[\mathcal{E}(x, \sigma_1)\ and\ \dots\ and\ \mathcal{E}(x, \sigma_k)] = \prod_{i=1}^{k} \mathbb{P}[\mathcal{E}(x, \sigma_i)]$$

$$N(\sigma) = \frac{k!}{d_1! \cdots d_k!}$$

In how many ways can a given type be distributed among bundles it originates from.

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \dots, \sigma_k)$.

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \prod_{i=1}^{k} \mathbb{P}[\mathcal{E}(x, \sigma_i)]$$

Probability that the exam paper x was ranked σ_i-th in a bundle can be further decomposed. How?

Idea: use the definition of noise matrix P

NOISE MATRIX REMINDER

Model students' **grading behaviour** by introducing a $k \times k$ **noise matrix** $P = (p_{i,j})_{i,j \in [k]}$, where $p_{i,j}$ denotes the probability that the student will rank the paper on the position i when its true rank is j.

Example: $p_{2,3} = 0.4$ means that the students will put the paper on the position 2 with probability 0.4 if its true ranking is 3.

FINDING $\mathbb{P}[\mathcal{E}(x, \sigma_i)]$

Intermediate problem of finding the probability that the exam x was ranked σ_i-th in a bundle.

Procedure:

- Consider all possible **true rankings** that an exam paper x may have in a bundle,
- Account for x having such a **true ranking** and in the same time being ranked σ_i-th by the grader (**noise matrix**!),

$$\mathbb{P}[\mathcal{E}(x, \sigma_i)] = \sum_{j=1}^{k} p_{\sigma_i, j} \binom{k-1}{j-1} x^{j-1}(1-x)^{k-j}$$

Sum over all possible **true rankings** of an exam paper in a bundle

FINDING $\mathbb{P}[\mathcal{E}(x, \sigma_i)]$

Intermediate problem of finding the probability that the exam x was ranked σ_i-th in a bundle.

Procedure:

- Consider all possible **true rankings** that an exam paper x may have in a bundle,
- Account for x having such a **true ranking** and in the same time being ranked σ_i-th by the grader (**noise matrix**!),

$$\mathbb{P}[\mathcal{E}(x, \sigma_i)] = \sum_{j=1}^{k} p_{\sigma_i, j} \binom{k-1}{j-1} x^{j-1}(1-x)^{k-j}$$

Probability of giving exam x rank σ_i if the **true rank** is j (**noise matrix**)

FINDING $\mathbb{P}[\mathcal{E}(x, \sigma_i)]$

Intermediate problem of finding the probability that the exam x was ranked σ_i-th in a bundle.

Procedure:

- Consider all possible **true rankings** that an exam paper x may have in a bundle,
- Account for x having such a **true ranking** and in the same time being ranked σ_i-th by the grader (**noise matrix**!),

$$\mathbb{P}[\mathcal{E}(x, \sigma_i)] = \sum_{j=1}^{k} p_{\sigma_i, j} \binom{k-1}{j-1} x^{j-1}(1-x)^{k-j}$$

Probability that the **true rank** of x in the bundle is j

FINDING $\mathbb{P}[\mathcal{E}(x, \sigma_i)]$

Intermediate problem of finding the probability that the exam x was ranked σ_i-th in a bundle.

Procedure:

- Consider all possible **true rankings** that an exam paper x may have in a bundle,
- Account for x having such a **true ranking** and in the same time being ranked σ_i-th by the grader (**noise matrix**!),

$$\mathbb{P}[\mathcal{E}(x, \sigma_i)] = \sum_{j=1}^{k} p_{\sigma_i, j} \binom{k-1}{j-1} x^{j-1} (1-x)^{k-j}$$

Number of ways in which papers ahead of **x** can be distributed in a bundle.

FINDING $\mathbb{P}[\mathcal{E}(x, \sigma_i)]$

Intermediate problem of finding the probability that the exam x was ranked σ_i-th in a bundle.

Procedure:

- Consider all possible **true rankings** that an exam paper x may have in a bundle,
- Account for x having such a **true ranking** and in the same time being ranked σ_i-th by the grader (**noise matrix**!),

$$\mathbb{P}[\mathcal{E}(x, \sigma_i)] = \sum_{j=1}^{k} p_{\sigma_i, j} \binom{k-1}{j-1} x^{j-1}(1-x)^{k-j}$$

Reminder: x is in range $[0,1]$ and the lower x, the better is its true ranking
Conclusion: $x^{j-1}(1-x)^{k-j}$ is the probability that there are $j-1$ papers in the bundle ahead of x.

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \ldots, \sigma_k)$.

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \prod_{i=1}^{k} \mathbb{P}[\mathcal{E}(x, \sigma_i)] = N(\sigma) \prod_{i=1}^{k} \sum_{j=1}^{k} p_{\sigma_i, j} \binom{k-1}{j-1} x^{j-1} (1-x)^{k-j}$$

Exchange sum and products operator and denoting by L_k set of **all** k-entry vectors $l = (l_1, \ldots, l_k)$ with $l_i \in \{1, \ldots, k\}$:

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \sum_{l \in L_k} \prod_{i=1}^{k} p_{\sigma_i, l_i} \binom{k-1}{l_i - 1} x^{l_i - 1} (1-x)^{k-l_i}$$

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \dots, \sigma_k)$.

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \sum_{l \in L_k} \prod_{i=1}^{k} p_{\sigma_i, l_i} \binom{k-1}{l_i - 1} x^{l_i - 1} (1-x)^{k - l_i}$$

Push multiplication to the exponent and denote $|l|_1 = \sum_{i=1}^{k} l_i$:

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \sum_{l \in L_k} \left(\prod_{i=1}^{k} p_{\sigma_i, l_i} \binom{k-1}{l_i - 1} \right) x^{|l|_1 - k} (1-x)^{k^2 - |l|_1}$$

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \dots, \sigma_k)$.

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Use $(1-x)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j x^j$

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \sum_{l \in L_k} \left(\prod_{i=1}^{k} p_{\sigma_i, l_i} \binom{k-1}{l_i - 1} \right) x^{|l|_1 - k} \sum_{j=0}^{k^2 - |l|_1} \binom{k^2 - |l|_1}{j} (-1)^j x^j$$

FINDING $\mathbb{P}[x \rhd \sigma]$

Now problem boils down to finding explicit formula for $\mathbb{P}[x \rhd \sigma]$, that is a probability that an exam paper x has type $\sigma = (\sigma_1, \dots, \sigma_k)$.

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \sum_{l \in L_k} \left(\prod_{i=1}^{k} p_{\sigma_i, l_i} \binom{k-1}{l_i - 1} \right) x^{|l|_1 - k} \sum_{j=0}^{k^2 - |l|_1} \binom{k^2 - |l|_1}{j} (-1)^j x^j$$

Order the terms:

$$\mathbb{P}[x \rhd \sigma] = N(\sigma) \sum_{l \in L_k} \sum_{j=0}^{k^2 - |l|_1} \left(\prod_{i=1}^{k} p_{\sigma_i, l_i} \binom{k-1}{l_i - 1} \right) \binom{k^2 - |l|_1}{j} (-1)^j x^{|l|_1 - k + j}$$

KEY Conclusion: $\mathbb{P}[x \rhd \sigma]$ is a univariate **polynomial** of degree $k^2 - k$

MODELLING WHOLE POPULATION FROM SAMPLE

MODELLING A POPULATION

Table III. Perfomance of the optimal type-ordering aggregation rules for approximations of the Mallows model. The data for Mallows are presented again here for direct comparison.

# samples	100		1000		Mallows	
setting	theory	$n = 10^4$	theory	$n = 10^4$	theory	$n = 10^4$
all2all	84.95	84.95	85.14	85.15	85.15	85.16
th-10%	91.82	91.85	92.05	92.04	92.05	92.07
th-50%	88.21	88.21	88.39	88.38	88.39	88.40
acc-2%	86.31	86.31	86.51	86.51	86.52	86.52
acc-5%	88.19	88.20	88.41	88.41	88.42	88.42

EXAMPLE ($k = 3$)

Very bad Graders

$$P = \begin{bmatrix} 0.1 & 0.3 & 0.6 \\ 0.3 & 0.4 & 0.3 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

REFORMULATE PROBLEM

$$\max_{\succ} \sum_{\sigma,\sigma' : \sigma \succ \sigma'} W(\sigma,\sigma') + \frac{1}{2}\sum_{\sigma} W(\sigma,\sigma)$$