

Exercise 6

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Network Decompositions

Exercise 1: Explain how given a $(\mathcal{C}, \mathcal{D})$ network decomposition of graph G , we can deterministically compute a $(\Delta + 1)$ -coloring of the graph in $O(\mathcal{C}\mathcal{D})$ rounds. Here, Δ denotes an upper bound on the maximum degree of the graph, and is given to the algorithm as an input.

Exercise 2: In this exercise, we prove that every n -node graph G has an $(\mathcal{C}, \mathcal{D})$ (strong-diameter) network decomposition for $\mathcal{C} = O(\log n)$ and $\mathcal{D} = O(\log n)$. The process that we see that be viewed as a simple and efficient sequential algorithm for computing such a network decomposition.

We determine the blocks G_1, G_2, \dots, G_C of network decomposition one by one, in C phases. Consider phase i and the graph $G \setminus (\cup_{j=1}^{i-1} G_j)$ remaining after the first $i - 1$ phases which defined the first i blocks G_1, \dots, G_{i-1} . To define the next block, we repeatedly perform a ball carving starting from arbitrary nodes, until all nodes of $G \setminus (\cup_{j=1}^{i-1} G_j)$ are removed. This ball carving process works as follows: consider an arbitrary node $v \in G \setminus (\cup_{j=1}^{i-1} G_j)$ and consider gradually growing a ball around v , hop by hop. In the k^{th} step, the ball $B_k(v)$ is simply the set all nodes within distance k of v in the remaining graph. In the very first step that the ball does not grow by more than a 2 factor — i.e., smallest value of k for which $|B_{k+1}(v)|/|B_k(v)| \leq 2$ — we stop the ball growing. Then, we carve out the inside of this ball — i.e., all nodes in $B_k(v)$ — and define them to be a cluster of G_i . Hence, these nodes are added to G_i . Moreover, we remove all boundary nodes of this ball — i.e., those of $B_{k+1}(v) \setminus B_k(v)$ — and from the graph considered for the rest of this phase. These nodes will never be put in G_i . We will bring them back in the next phases, so that they get clustered in the future phases. Then, we repeat a similar ball carving starting at an arbitrary other node v' in the remaining graph. We continue a similar ball carving until all nodes are removed. This finishes the description of phase i . Once no node remains in this graph, we move to the next phase. The algorithm terminates once all nodes have been clustered.

Prove the following properties:

1. Each cluster defined in the above process has diameter at most $O(\log n)$. In particular, for each ball that we carve, the related radius k is at most $O(\log n)$.
2. In each phase i , the number of nodes that we cluster — and thus put in G_i — is at least $1/2$ of the nodes of $G \setminus (\cup_{j=1}^{i-1} G_j)$.
3. Conclude that the process terminates in at most $O(\log n)$ phases, which means that the network decomposition has at most $O(\log n)$ blocks.

Exercise 3 (optional): Develop a deterministic distributed algorithm with round complexity $\text{poly}(\log n)$ for computing an $(\mathcal{C}, \mathcal{D})$ strong-diameter network decomposition in any n -node network, such that $\mathcal{C} = O(\log n)$ and $\mathcal{D} = O(\log n)$.