Bonus Exercise

April 2020

Zero-One Principle

Suppose that you are given a comparison network that transforms the input sequence $a = (a_1, a_2, \dots, a_n)$ into the output sequence $b = (b_1, b_2, \dots, b_n)$. In addition, suppose you are given a monotonically increasing function $f : \mathbb{N}^n \mapsto \mathbb{N}^n$. Note that a function f is called monotonically increasing if for all fixed $(a_1, a_2, \dots, a_n), (a'_1, a'_2, \dots, a'_n) \in \mathbb{N}^n$, $(a_1 \leq a'_1$ and $a_2 \leq a'_2$ and $\dots$ and $a_n \leq a'_n) \Rightarrow f(a_1, a_2, \dots, a_n) \leq f(a'_1, a'_2, \dots, a'_n)$.

A) Prove that a single comparator with inputs $f(x), f(y) \in \mathbb{N}$ produces the outputs $f(\max(x, y))$ and $f(\min(x, y))$.

B) Prove that the comparison network transforms the input sequence $f(a) = (f(a_1), f(a_2), \dots, f(a_n))$ into the output sequence $f(b) = (f(b_1), f(b_2), \dots, f(b_n))$.

C) Use question **B)** to prove the following (*0-1 sorting lemma*): If a comparison network with n inputs sorts all 2^n possible sequences of 0's and 1's correctly, then it sorts all sequences of arbitrary numbers correctly.

Solution

A) Suppose we apply $f(x)$ and $f(y)$ to the inputs of the comparator, such that $f(x) \leq f(y)$, wlog. The operation of the comparator yields the value $\min(f(x), f(y)) = f(x)$ on the upper output and the value $\max(f(x), f(y)) = f(y)$ on the lower output. Since f is monotonically increasing, $f(x) \leq f(y) \implies x \leq y$ and thus $f(max(x, y)) = f(y) = \max(f(x), f(y))$. Similarly, $f(min(x, y)) = f(x) = \min(f(x), f(y))$.

B) We use induction on the depth of each wire in a general comparison network to prove the following statement: if a wire has the value a_i when the input sequence a is applied to the network, then it has the value $f(a_i)$ when the input sequence $f(a)$ is applied. This holds also for the output wires thus it proves the statement. For the basis, consider a wire at depth 0, that is, an input wire a_i. Obviously, when $f(a)$ is applied to the network, the input wire has the value $f(a_i)$. For the inductive step, consider a wire at depth d, where $d \geq 1$. The wire is the output of a comparator at depth d, and the input wires to this comparator are at a depth strictly less than d. By the inductive hypothesis, if the input wires to the comparator have values a_i and a_j when the input sequence a is applied, then they have $f(a_i)$ and $f(a_j)$ when the input sequence $f(a)$ is applied. From question **A)** we know that the output wires of this comparator then have $f(min(a_i, a_j))$ and $f(max(a_i, a_j))$.

C) [Towards contradiction.] Suppose the comparison network sorts all $0-1$ sequences, but there exists a sequence of arbitrary numbers that the network does not sort correctly. Let this sequence be $a = (a_1, a_2, \ldots, a_n)$, containing a_i and a_j such that $a_i < a_j$ while the network's output sequence places a_j before a_i. We define the following monotonically increasing function:

$$f(x) = \begin{cases} 0, & \text{if } x \leq a_i \\ 1, & \text{if } x > a_j \end{cases}$$

Since the network's output sequence places a_j before a_i when $a = (a_1, a_2, \ldots, a_n)$ is the input, it follows from question **B)** that it places also $f(a_i)$ before $f(a_j)$ in the output sequence when $(f(a_1), f(a_2), \ldots, f(a_n))$ is input. However, $f(a_j) = 1$ and $f(a_i) = 0$, and thus the comparison network fails to sort the $0 - 1$ sequence $(f(a_1), f(a_2), \ldots, f(a_n))$ correctly. Contradiction.