HS 2008 Prof. Dr. R. Wattenhofer / Raphael Eidenbenz / Roland Flury

Discrete Event Systems
Exercise 10

1 The Winter Coat Problem

While exploring the sea with his boat, Mr. Robinson lost orientation and ended up on a strange island. After living there for several years, he observed that the weather on the island follows a strict probabilistic pattern: The weather of a given day only depends on the weather the day before. When it was sunny (S), the following day is sunny again with 50%, but it gets cloudy (C) with 30% and it starts to rain (R) with 20%. Once its cloudy, it says cloudy with 10%, gets sunny with 70% and starts to rain with 20%. Finally, after a rainy day, it gets sunny with 10%, cloudy with 40% and remains rainy with 50%.

a) Model this special weather condition using a Markov chain.

b) After spending a sunny day, how many days does Mr. Robinson have to wait until its sunny again (in expectation)?

Due to the global warming of earth, the weather conditions are actually slightly different: After a sunny day, it remains sunny only with 49%, but gets hot (H) with 1%. Once it's hot, it remains hot forever. Similarly, after a rainy day, it remains rainy with 49%, but starts to snow (W) with 1%. Again, once it snows, it continues to snow forever.

c) Adapt your Markov chain to model the new situation.

d) What is the probability that Mr. Robinson ever needs a winter coat, given that he arrived on a sunny day on the island?

2 Probability of Arrival

In the script, there is a lemma saying that the probability of arrival can be computed as

$$f_{ij} = p_{ij} + \sum_{k:k \neq j} p_{ik} f_{kj}.$$

Prove this lemma.

3 Night Watch

In order to improve their financial situation, Raphael and Roland also work at nights. Their task is to guard a famous Swiss bank which, from a architectonic perspective, looks as follows:

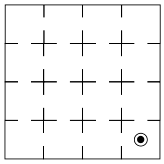

Figure 1: Offices of a Swiss bank.

Thus, there are 4x4 rooms, all connected by doors as indicated in the figure.

In a first scenario, Raphael and Roland always stay together. They start in the room on the upper left. Every minute, they change to the next room, which is chosen uniformly at random from all possible (adjacent) rooms.

a) Compute the probability (in the steady state) that Roland and Raphael are in the room where the thief enters the bank (indicated with $\odot$)!

b) Since Roland and Raphael are very strong, they can easily catch a thief on their own. Thus, in a second scenario, they decide that it be smarter to patrol individually: After every minute, each of them chooses the next room *independently*. What is now the probability that at least one of them is in the room where the thief enters?

Note: This exercise is inspired by an exam question of the winter term 2004/2005.

4 "Hopp FCB!"

Besides its moodiness, the *FC Basel* (FCB) soccer club is confronted with yet another problem: sporadically, its players get sick. Assume that the whole team consists of n players. Assume that the time period a fit player remains fit is exponentially distributed with parameter μ (independently of the state of the other players). On the other hand, the time a sick player remains sick is exponentially distributed with parameter λ.

a) Model the situation as a birth-and-death Markov process where the states denote the number of players which are fit.

b) Derive a formula for the probability that exactly i players are fit.

c) Assume that the FCB has 20 players, and that $\lambda^{-1} = 4$ months and $\mu^{-1} = 10$ months. Calculate the probability that the FCB can participate at a given match.