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HS 2009
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Discrete Event Systems
Sample Solution to Exercise 4

1 Regular and Context-Free Languages

a) Sometimes, even simple grammars can produce tricky languages. We can interpret the 1s and 2s of the second production rule as opening and closing brackets. Hence, $L(G)$ consists of all correct bracket terms where at least one 0 must be in each bracket.

Choose $w = 1^p 0 2^p \in L(G)$. Let $w = xyz$ with $|xy| \le p$ and $|y| > 0$ (pumping lemma). Because of $|xy| \le p$, xy is in the first 1^p of w. According to the pumping lemma, we should have $xy^iz \in L (i \ge 0)$. However, by choosing $i = 0$ we delete at least one 1 and get a word $w' = 1^k 0 2^p$ where $k < p$. w' is not in L since it has less 1s than 2s. w is not pumpable. Therefore $L(G)$ is not regular.

b) Since *every* regular language is also context-free, we can choose an arbitrary regular language. For example, we can choose the language $L = \{0^n 1, n \ge 1\}$ which is clearly regular. The corresponding context-free grammar is $S \to 0S \mid 01$.

2 Context-Free Grammars

a)

$$
\begin{aligned}
S &\to SAS \mid A \\
A &\to 0 \mid 1
\end{aligned}
$$

Note: The language is regular!

b) One possible solution is to use three productions: A first one which guarantees that there is at least one '1' more; a second one which produces all possible strings with the same number of '0' and '1'; and finally, a production to add further 1's at arbitrary places:

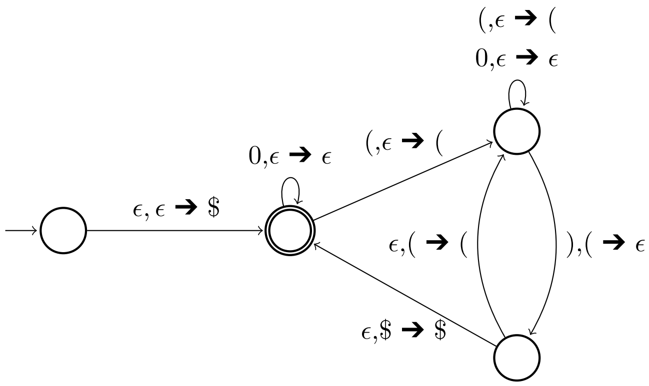
$$
\begin{aligned}
S &\to T1T \\
T &\to T0T1T \mid T1T0T \mid U \\
U &\to 1U \mid \epsilon
\end{aligned}
$$

3 Pushdown Automata

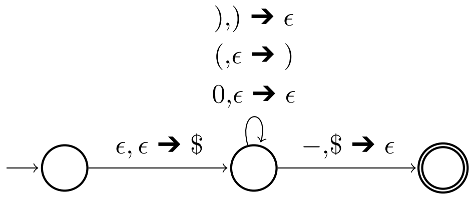
a) ϵ, 0, 00, ()

b) It is unambiguous, i.e., there is a unique derivation tree for each word. Each word $w \neq \epsilon$ in $L(G)$ contains a rightmost 0 or parenthesis expression $'(S)'$ that can be unanimously assigned to a A in each node of the derivation tree. Due to $S \to SA$, each sequence of As is unambiguous.

c) A push-down automaton M is *deterministic* iff in each state, there is exactly one successor state for any combination $(x, y) \in \Sigma \times \Gamma$ where Σ is the string input alphabet and Γ is the stack alphabet. Note that if a state q has only one outgoing transition 'ϵ,ϵ ➔ \$' the PDA is still deterministic since there is no ambiguity of what the successor state of q will be. If a state q, however, has two outgoing transitions, 'ϵ,\$ ➔ \$' and '(,ϵ ➔ \$', it is unclear which transition the system should take if the string input in state q is '(' and the top element on the stack is ‘\$’. As with deterministic FAs we take the liberty of omitting transitions leading to an (imaginary) fail state as well as the fail state itself when drawing deterministic PDAs. An instance of a deterministic PDA accepting $L(G)$ is the following:



If we would assume our PDA recognizes the end of the input string (denoted by '$-$') the following deterministic pushdown automaton would also do the job:



Note that by replacing '$-$' in the above PDA by 'ϵ' we get a correct non-deterministic PDA for $L(G)$.

4 Pumping Lemma revisited

a) Let us assume that L is regular and show that this results in a contradiction.

We have seen that any regular language fulfills the pumping lemma. I.e. there is a p, such that for every word $u \in L$ with $|u| \geq p$ it holds that: u can be written as $u = xyz$ with $|xy| \leq p$ and $1 \leq |y| \leq p$, such that $\forall i \geq 0 : xy^iz \in L$.

In order to obtain the contradiction, we need to show that there is at least one word $w \in L$ with $|w| \geq p$ for which it is not possible to form the string partition $w = xyz$, s.t. $|xy| \leq p$, $1 \leq |y| \leq p$, and $\forall i \geq 0 : xy^iz \in L$.

First, we need to overcome the problem that we do not know the value of p. The standard trick is to consider words whose length depends on p. E.g. consider the word $w = 1^{p^2} \in L$. For sure, $|w| \geq p$.

By the pumping lemma, we can write $w = 1^{p^2}$ as xyz. What remains to show is that there is no partition xyz that satisfies $|xy| \leq p$, $1 \leq |y| \leq p$, and $\forall i \geq 0 : xy^iz \in L$.

The expression $w = xy^iz$ can be written as $xy^iz = 1^{|x|}1^{i|y|}1^{|z|}$. Because $|w| = p^2$, we know that $|z| = p^2 - |x| - |y|$, and therefore, $xy^iz = 1^{|x|}1^{i|y|}1^{p^2-|x|-|y|} = 1^{p^2+(i-1)|y|}$.

To obtain the contradiction, we need to find an $i \geq 0$, such that $xy^iz \notin L$. For example, consider $i = 0$. Then we have $w^0 = xy^0z = 1^{p^2-|y|}$. Clearly, $|w^0| < p^2$, as $|y| \geq 1$. Note that we argue independent of the partition $w = xyz$, we do not pick a specific x and y and therefore the following holds for all possible partitions.

If $w^0 \in L$, then $|w^0|$ is a square number, smaller than p^2. But the next smaller square number, $(p-1)^2$, is strictly smaller than $|w^0|$: $(p-1)^2 = p^2 - 2p + 1 < p^2 - p \leq p^2 - |y| = |w^0|$, which shows that $|w^0|$ cannot be a square number. This shows that there is *no* partition for w that allows to fulfill the pumping lemma conditions. But this should be the case if L is regular. Thus, we have a contradiction, which concludes the proof.

b) Consider the alphabet $\Sigma = \{a_1, a_2, ..., a_n\}$ and the language $L = \bigcup_{i-1}^{n} a_i^*$. The language is regular, as it is the union of regular languages, and the smallest possible pumping number p for L is 1. But any DFA needs at least $n+1$ states to distinguish the n different characters of the alphabet. Thus, for the DFA, we cannot deduce any information from p about the minimum number of states.

The same argument holds for the NFA.