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Discrete Event Systems
Solution to Exercise 7

1 Probability of Arrival

The proof is similar to the one about the transition time h_{ij} (see script). We express f_{ij} as a condition probability that depends on the result of the first step in the Markov chain. Recall that the random variable T_{ij} is the *hitting time*, that is, the number of steps from i to j. We get $Pr[T_{ij} < \infty \mid X_1 = k] = Pr[T_{ij} < \infty]$ for $k \neq j$ and $Pr[T_{ij} < \infty \mid X_1 = j] = 1$. We can therefore write f_{ij} as

$$\begin{aligned} f_{ij} &= Pr[T_{ij} < \infty] = \sum_{k \in S} Pr[T_{ij} < \infty \mid X_1 = k] \cdot p_{ik} \\ &= p_{ij} \cdot Pr[T_{ij} < \infty \mid X_1 = j] + \sum_{k \neq j} Pr[T_{ij} < \infty \mid X_1 = k] \cdot p_{ik} \\ &= p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}. \end{aligned}$$

2 Basketball [Exam]

a) This exercise is a good example to illustrate the fact that most exercises allow several ways to be solved.

Variant A. As Mario scores with probability p in each attempt Mario scores an expected p hits in one attempt. Due to the linearity of the expected value of independent random variables we may now compute the expected number $E[X]$ of attempts until he succeeded m times by the following equation:

$$E[X] \cdot p = m \Longleftrightarrow E[X] = \frac{m}{p}.$$

After $\frac{m}{p}$ attempts, Mario has scored an expected m hits and he has missed $\frac{m}{p} - m$ times. Hence he does an expected $10(\frac{m}{p} - m)$ push-ups in the game.

Variant B. We define a random variable X that counts the number of successful attempts before the first miss. X is distributed as follows:

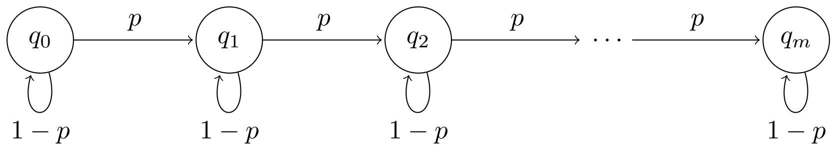
$$\begin{aligned} Pr[X = 0] &= 1 - p \\ Pr[X = 1] &= p(1 - p) \\ Pr[X = 2] &= p^2(1 - p) \\ &\vdots \\ Pr[X = i] &= p^i(1 - p) \end{aligned}$$

We get for the expected value of X,

$$E[X] = \sum_{x \in X} x \cdot Pr[X = x] = \sum_{i=0}^{\infty} i \cdot p^i(1 - p) = (1 - p)\frac{p}{(1 - p)^2} = \frac{p}{1 - p}$$

Again, due to the linearity of the expected value we may think of the game as Mario scoring $E[X]$ hits, missing once, scoring the next $E[X]$ hits, missing again, and so forth until he scored a total of m hits. The question of how often Mario misses now translates to the question of how many series of $E[X]$ successful attempts he needs in order to score m times, and we get $10\frac{m}{E[X]} = 10(\frac{m}{p} - m)$ push-ups in expectation.

Variant C (Markov Chain). The following Markov chain models Mario's game.



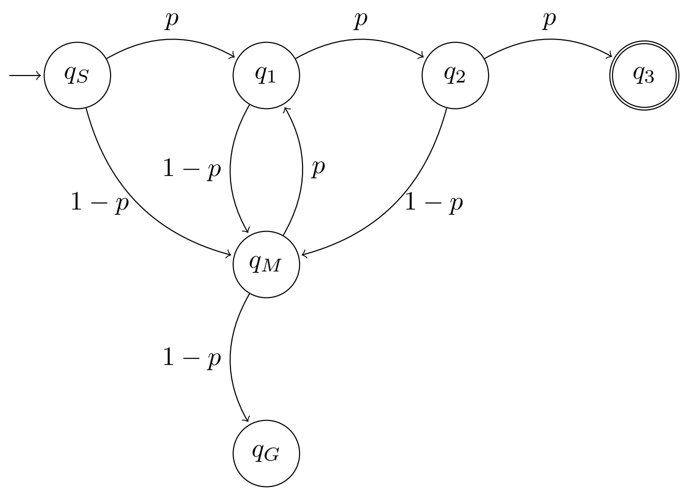
In a state q_i Mario has scored i hits. To learn the expected number of attempts until Mario has scored m hits we can simply compute the hitting time h_{0m} from q_0 to q_m.

$$
\begin{aligned}
h_{0m} &= 1 + \sum_{k \neq m} p_{0k} h_{km} = 1 + p_{00} h_{0m} + p_{01} h_{1m} \\
h_{0m} &= \frac{1 + p_{01} h_{1m}}{1 - p_{00}} = \frac{1 + p h_{1m}}{p} = \frac{1}{p} + h_{1m} \\
h_{1m} &= 1 + p_{11} h_{1m} + p_{12} h_{2m} \iff h_{1m} = \frac{1}{p} + h_{2m} \\
h_{0m} &= \frac{1}{p} + h_{1m} = \frac{2}{p} + h_{2m} = \ldots = \frac{m}{p} + h_{mm} = \frac{m}{p}
\end{aligned}
$$

By subtracting the m successful attempts, we get an expected $\frac{m}{p} - m$ misses and hence Mario does $10(\frac{m}{p} - m)$ push-ups in expectation.

b) Luigi's probability of scoring m subsequent hits is p^m. In expectation he needs $1/p^m$ attempts until he succeeds. In the last attempt he does not do any push-ups, hence we expect him to do $10 \cdot (\frac{1}{p^m} - 1)$ push-ups.

c) The following Markov chain models Trudy's game.



In state q_i Trudy has scored i hits in a row, in q_M she has missed once, in q_G she has missed twice in a row and gives up.

(i) We determine the probability f_{S3} of reaching the accepting state q_3 from the start state q_S.

$$
\begin{aligned}
f_{S3} &= p \cdot f_{13} + (1-p) \cdot f_{M3} \\
f_{13} &= p \cdot f_{23} + (1-p) \cdot f_{M3} \\
f_{23} &= p + (1-p) \cdot f_{M3} \\
f_{M3} &= p \cdot f_{13}
\end{aligned}
$$

$$
\begin{aligned}
f_{13} &= p^2 + (1-p)p^2 \cdot f_{13} + (1-p)p \cdot f_{13} \\
&= \frac{p^2}{1+p^3-p} = 0.4
\end{aligned}
$$

$$
\begin{aligned}
f_{S3} &= p \cdot \frac{p^2}{1+p^3-p} + (1-p)p \cdot \frac{p^2}{1+p^3-p} \\
&= \frac{2p^3-p^4}{1+p^3-p} \\
&\approx 0.3
\end{aligned}
$$

The probability that Trudy scores 3 times in a row is 0.3. The probability that she gives up is 0.7. This is because q_3 and q_G are the only absorbing states, i.e., all other states have probability mass of 0 in the steady state.

(ii) To get the number of push-ups we define a random variable Z that counts how often the system passes state q_M before either ending up in state q_M or in state q_G. E.g., the probability $P[Z=1]$ of passing q_M exactly once equals the probability of getting from q_S to q_M without being absorbed by q_3 and then ending up directly in q_G or q_3, i.e. $Pr[Z=1] = P_{SM} \cdot (P_{MG} + P_{M3})$ where P_{ij} is the probability of getting from q_i to q_j without passing q_M on the way. Z has the following probability distribution:

$$
\begin{aligned}
Pr[Z=0] &= 1 - P_{SM} \\
Pr[Z=1] &= P_{SM} \cdot (P_{MG} + P_{M3}) \\
Pr[Z=2] &= P_{SM} \cdot P_{MM} \cdot (P_{MG} + P_{M3}) \\
Pr[Z=3] &= P_{SM} \cdot P_{MM}^2 \cdot (P_{MG} + P_{M3}) \\
&\vdots \\
Pr[Z=i] &= P_{SM} \cdot P_{MM}^{i-1} \cdot (P_{MG} + P_{M3})
\end{aligned}
$$

The probability of passing q_M exactly i times equals the probability of getting from q_S to q_M and from q_M to q_M again $i-1$ times and then ending up directly in q_G or q_3. As the Markov chain is not too complicated we can compute the needed P_{ij} rather easily and get $P_{SM} = 1-p^3$, $P_{MM} = p - p^3$, $P_{MG} = 1-p$, and $P_{M3} = p^3$.

The expected number of misses is

$$
\begin{aligned}
E[Z] &= \sum_{i=1}^{\infty} i \cdot Pr[Z=i] \\
&= \sum_{i=1}^{\infty} i \cdot P_{SM} \cdot P_{MM}^{i-1} \cdot (P_{MG} + P_{M3}) \\
&= P_{SM} \cdot (P_{MG} + P_{M3}) \cdot \sum_{i=1}^{\infty} i \cdot P_{MM}^{i-1} \\
&= \frac{P_{SM} \cdot (P_{MG} + P_{M3})}{(1-P_{MM})^2} \\
&= \frac{(1-p^3) \cdot (1-p+p^3)}{(1-p+p^3)^2} = \frac{1-p^3}{1-p+p^3} \\
&= \frac{1-\frac{1}{8}}{1-\frac{1}{2}+\frac{1}{8}} = \frac{7}{5} = 1.4.
\end{aligned}
$$

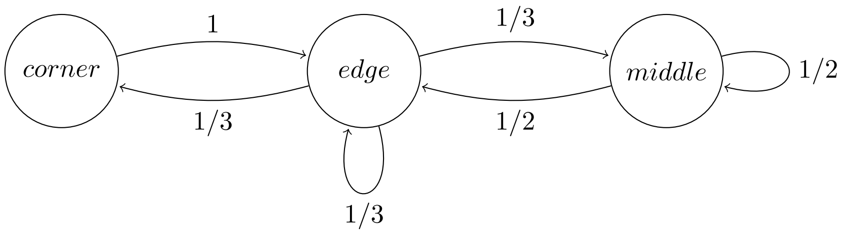
Hence, Trudy does 14 push-ups in expectation.

Variant. We already know that Trudy gives up with a probability 0.7. Each time Trudy is in q_M she gets to q_G with probability $1-p$. Hence it must hold that $E[Z] \cdot (1-p) = 0.7$. This yields for the expected number of push-ups

$$
10 \cdot E[Z] = 10 \cdot \frac{0.7}{1-p} = 10 \cdot 2 \cdot 0.7 = 14.
$$

3 Night Watch

a) Observe that the problem is symmetric, e.g., from all four corners, the situation looks the same, and the probability of being in a specific corner room is the same for all corners. The same holds for rooms at the border and for rooms in the middle. Thus, instead of using 16 states, we consider the following simplified Markov chain consisting of 3 states only:



Hence, in the steady state, it holds that

$$P_c = 1/3 \cdot P_e; \quad P_e = 1/3 \cdot P_e + 1/2 \cdot P_m + P_c; \quad 1 = P_c + P_e + P_m$$

Solving this equation system gives: $P_c = 1/6$. The probability of being in a specific corner is therefore $1/6 \cdot 1/4 = 1/24$.

b) Since the two walks are independent, we have

$$1/24 + 1/24 - (1/24)^2 = 0.082.$$