HS 2009

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Discrete Event Systems
Solution to Exercise 8

1 Gloriabar

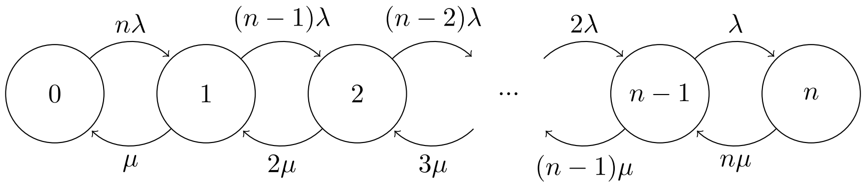
a) The situation can be modeled by a M/M/1 queue. We have an arrival rate of $\lambda = 540/(90 \cdot 60) = 1/10$ (persons per second), and $\mu = 1/9$ (persons per second). Thus $\rho = \lambda/\mu = 9/10$. We can apply Little's Law (slides 76 ff.) and therefore, we can use the formulae for the response and waiting time from slide 79: The expected waiting time is $W = \rho/(\mu - \lambda) = 81$ seconds. The expected time until the student has paid for her menu is given by $T = 1/(\mu - \lambda) = 90$ seconds.

b) We use the formula for the expected number of jobs in the queue from slide 79 and obtain queue length of $N = \rho^2/(1 - \rho) = 8.1$.

c) We require that $T = 1/(\mu - 0.1) = 90/2$. Thus, $\mu = 11/90$, i.e., instead of 9 secs, the service time should be $90/11 \approx 8.2$ secs.

2 "Hopp FCB!"

a) We know that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$. Thus, we have the following birth-death-process:



b) Let π_i be the probability of state i in the equilibrium. From slide 87, we know that

$$\pi_i = \pi_0 \cdot \prod_{j=0}^{i-1} \frac{\lambda_j}{\mu_{j+1}}$$

and thus

$$\pi_i = \pi_0 \cdot \frac{\lambda_0 \cdot \lambda_1 \cdot \dots \cdot \lambda_{i-1}}{\mu_1 \cdot \mu_2 \cdot \dots \cdot \mu_i}.$$

Applying this formula to our process yields

$$\pi_i = \pi_0 \cdot \frac{n(n-1) \cdot \dots \cdot (n-i+1) \cdot \lambda^i}{1 \cdot 2 \cdot \dots \cdot i \cdot \mu^i} = \pi_0 \cdot \binom{n}{i} (\rho)^i$$

where $\rho := \frac{\lambda}{\mu}$. We know that the sum of all probabilities equals 1, so we have

$$\sum_{i=0}^{n} \pi_i = \pi_0 \sum_{i=0}^{n} \binom{n}{i} \rho^i = 1$$
$$\Leftrightarrow \quad \pi_0 (1+\rho)^n = 1 \tag{1}$$

For conversion (1) we used the formula for the binomial series:

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i.$$

Finally, we obtain

$$\pi_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n}.$$

c) A team is able to play if and only if there are at least eleven fit players:

$$\pi_{11} + \pi_{12} + \cdots + \pi_{20} = 0.965.$$

Thus, the FCB team has enough players that it can participate in most of the matches (probability > 95 %).

3 Theory of Ice Cream Vending

The situation can be described by a classic M/M/2 system. According to slide 90, there is an equilibrium iff

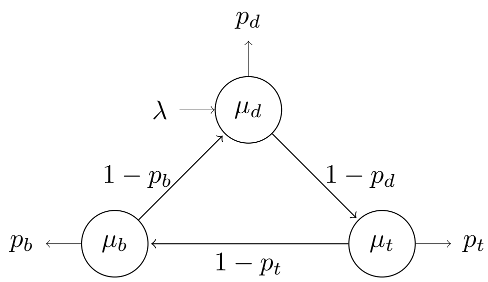
$$\rho = \lambda/(2\mu) < 1.$$

For the stationary distribution, it holds that

$$\pi_0 = \frac{1}{1 + 2\rho + 4\rho^2/(2(1-\rho))} = \frac{1-\rho}{1+\rho}.$$

4 Queuing Networks

a)



b) We have an open queuing network and hence we can apply Jackson's theorem (slides 97ff):

$$\lambda_d = \lambda + \lambda_b(1-p_b)$$
$$\lambda_t = \lambda_d(1-p_d)$$
$$\lambda_b = \lambda_t(1-p_t)$$

Solving this equation system gives:

$$\lambda_d = \frac{\lambda}{1-(1-p_d)(1-p_t)(1-p_b)}$$
$$\lambda_t = \frac{(1-p_d)\lambda}{1-(1-p_d)(1-p_t)(1-p_b)}$$
$$\lambda_b = \frac{(1-p_d)(1-p_t)\lambda}{1-(1-p_d)(1-p_t)(1-p_b)}$$

c) The waiting time is given by $W_t = \rho_t/(\mu_t - \lambda_t)$, where $\rho_t = \lambda_t/\mu_t$.

d) We apply the given values to the equations for λ_d, λ_t and λ_b and obtain:

$$\lambda_d = 10, \qquad \lambda_t = 25/3, \qquad \lambda_b = 20/3.$$

Therefore, by the formula of slide 73, the expected number of customers in the system is given by

$$N = \frac{\lambda_d}{\mu_d - \lambda_d} + \frac{\lambda_t}{\mu_t - \lambda_t} + \frac{\lambda_b}{\mu_b - \lambda_b} = 8.$$

Applying Little's formula to the entire system gives $T = N/\lambda = 8/5$ hours.

e) We require $\lambda_t = 1$ and therefore

$$\frac{\lambda(1 - p_d)}{1 - (1 - p_d)(1 - p_t)(1 - p_b)} = 1.$$

Solving the equation for p_d yields:

$$p_d = 1 - \frac{1}{\lambda + (1 - p_t)(1 - p_b)} = 1 - \frac{1}{5 + \frac{4}{5} \cdot \frac{3}{4}} = 1 - \frac{1}{\frac{28}{5}} = \frac{23}{28}.$$