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Discrete Event Systems

Solution to Exercise Sheet 11

1 Competitive Christmas

a) If our algorithm tells Roger to buy a tree of size c, then it is followed in the worst-case by a tree of size b which yields a competitive ratio of $\rho_A = b/c$. To make ρ_A as small as possible, Roger should only buy a tree if its size c is as large as possible. If he ignores a tree of size $c-\varepsilon$ in the hope that he finds a larger tree, he might in the worst-case only encounter trees of size a yielding a competitive ratio of $\rho_B = (c-\varepsilon)/a$. In these two cases ρ should be as small as possible such that Roger gets a tree that is as large as possible. Hence, we have to set the value c above which Roger buys a tree such that ρ_A and ρ_B are minimised. This is the case for $\rho_A = \rho_B$ which yields $c = \sqrt{ab}$. The corresponding competitive ratio is $\sqrt{b/a}$.

b) Discussed in **a)**.

2 The Best Secretary

a) To avoid a potentially bad ordering of the candidates, Roger invites them into his room in a random order. Within the first half ($n/2$ candidates), Roger only rates the applicants and memorises the best quality G. In the second half, Roger accepts a candidate if its quality exceeds G.

The probability that the second-best candidate is in the first half is $1/2$. The probability that the best candidate is in the second half is also $1/2$. If both these events happen, Roger actually hires the best candidate. This case occurs with probability $1/2 \cdot 1/2 = 1/4$.

b) The analysis for the above algorithm clearly is only a rough estimate. We only considered the special case where the second-best secretary is in the first half and the best one in the second half. Roger also would have chosen the best one if the third-best is in the first half and the best and the second-best candidate in the second half but in this order. Furthermore, we assumed without justification that the algorithm rates the *first half* of the candidates and then selects one from the second half.

For an exact analysis, assume that Roger first rates $x \cdot n$ candidates (for $x \in [0,1]$) and then chooses the candidate that is better than the best one from these $x \cdot n$ applicants. Let **1** and **2** denote the set of the candidates in the first and second part, respectively.

We denote by C_i the i-th best candidate and define the probability p_i for the event that $C_i \in \mathbf{1}$ and all better ones are in **2** with the best candidate first. The case (special case from above) that the $C_2 \in \mathbf{1}$ and $C_1 \in \mathbf{2}$ occurs with probability

$$p_2 = \Pr[C_2 \in \mathbf{1}] \cdot \Pr[C_1 \in \mathbf{2} \mid C_2 \in \mathbf{1}]$$
$$= x \cdot \frac{(1-x) \cdot n}{n-1} \quad .$$

Note that the conditional probability for C_1 being in **2** given that C_2 is in **1** is bigger than $(1-x)$ since one "slot" in **1** is already taken by C_2.

Similiarly, we can calculate the probability p_3 but we now also have to consider the probability that C_1 appears before C_2 in **2**.

$$\begin{aligned} p_3 &= \Pr[C_3 \in \mathbf{1}] \cdot \Pr[C_2 \in \mathbf{2} \mid C_3 \in \mathbf{1}] \cdot \Pr[C_1 \in \mathbf{2} \mid C_3 \in \mathbf{1} \text{ and } C_2 \in \mathbf{2}] \cdot \Pr[C_1 \text{ before } C_2] \\ &= x \cdot \frac{(1-x)\cdot n}{n-1} \cdot \frac{(1-x)\cdot n - 1}{n-2} \cdot \frac{1}{2} \end{aligned}$$

Note again that we have to use conditional probabilities here.

Analogous to p_3, we can derive a formula for p_i.

$$\begin{aligned} p_i &= x \cdot \left(\prod_{j=0}^{i-2} \frac{(1-x)\cdot n - j}{n-j-1} \right) \cdot \frac{1}{i-1} \\ &= x \cdot \left(\prod_{j=0}^{i-2} (1-x) \cdot \left(1 + \frac{1}{n-j-1}\right) \right) \cdot \frac{1}{i-1} \end{aligned}$$

Since we are interested in the probabilities for large n, we can consider the limit instead.

$$\begin{aligned} p_i' = \lim_{n\to 0} p_i &= \lim_{n\to 0} x \cdot \left(\prod_{j=0}^{i-2} (1-x) \cdot \left(1 + \frac{1}{n-j-1}\right) \right) \cdot \frac{1}{i-1} \\ &= x \cdot \left(\prod_{j=0}^{i-2} (1-x) \right) \cdot \frac{1}{i-1} \\ &= x \cdot (1-x)^{i-1} \cdot \frac{1}{i-1} \end{aligned}$$

Note that this result corresponds to ignoring the dependence between the candidates in both parts by calculating without conditional probabilities.

Now we can calculate the success probability p_{succ} for hiring the best candidate.

$$\begin{aligned} p_{\text{succ}}(x) &= \sum_{i=2}^{\infty} p_i' \\ &= \sum_{i=1}^{\infty} \frac{x}{i} \cdot (1-x)^i \\ &= x \cdot \sum_{i=1}^{\infty} \frac{(1-x)^i}{i} \\ &= -x \ln(x) \end{aligned}$$

Differentiating yields that $p_{\text{succ}}(x)$ attains its maximum for $x = 1/e$ and we have further $p_{\text{succ}}(1/e) = 1/e$. Hence, Roger should only rate a fraction of $1/e \approx 37\,\%$ of the candidates and then start with the selection as described above. Then, the probability to get the best secretary is $1/e$.