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Discrete Event Systems

Solution to Exercise Sheet 13

1 Power-Down Mechanisms

As mentioned in the hint, we only focus on a single idle period because if we know that our algorithm is c-competitive for any idle period, we also know that it is c-competitive for the complete busy sequence.

a) Analogously to the 2-competitive ski-rental online algorithm, we consider an algorithm ALG that powers down after D time units. To see that ALG is 2-competitive, we distinguish two cases for the length of the current idle period T:

- $T < D$: The energy consumed by both algorithms is $c_{\text{ALG}} = c_{\text{OPT}} = T$, hence the competitive ratio is $c = T/T = 1$.
- $T \geq D$: We have $c_{\text{ALG}} = D + D$ since ALG waits D time units and then powers-down and $c_{\text{OPT}} = D$ because OPT powers down immediately. Hence we get

$$c = \frac{2D}{D} = 2 \ .$$

b) Let ALG be any *deterministic* power down algorithm. Then the time t_{ALG} after which it powers down in an idle period is known in advance. The "worst" idle period ends immediately after ALG has powered down, that is we have $T = t_{\text{ALG}} + \varepsilon$. Again, we distinguish two cases with respect to the time t_{ALG} when ALG powers down.

- $t_{\text{ALG}} < D$: We have $c_{\text{ALG}} = t_{\text{ALG}} + D$ and $c_{\text{OPT}} = t_{\text{ALG}} + \varepsilon$, hence

$$c = \frac{t_{\text{ALG}} + D}{t_{\text{ALG}} + \varepsilon} = 1 + \frac{D - \varepsilon}{t_{\text{ALG}} + \varepsilon} > 2 \quad \text{for } \varepsilon \to 0$$

  since $t_{\text{ALG}} < D$.
- $t_{\text{ALG}} \geq D$: We have $c_{\text{ALG}} = t_{\text{ALG}} + D$ again and $c_{\text{OPT}} = D$, hence

$$c = \frac{t_{\text{ALG}} + D}{D} = 1 + \frac{t_{\text{ALG}}}{D} \geq 2 \quad \text{for } \varepsilon \to 0$$

  since $t_{\text{ALG}} \geq D$.

Hence, ALG cannot be better than 2-competitive.

c) Let ALG be a randomised algorithm that powers down at time $\frac{2}{3}D$ with probability $\frac{1}{2}$ and at time D otherwise. Let C_{ALG} be a random variable for the cost incurred by the algorithm. We again consider an arbitrary idle period of length T. We distinguish three cases:

- $T < \frac{2}{3}D$: The energy consumption of both algorithms is $c_{\text{ALG}} = c_{\text{OPT}} = T$, hence $c = T/T = 1 < 2$.

- $\frac{2}{3}D \leq T < D$: The expected energy consumption of ALG is

$$\mathbf{E}[C_{\text{ALG}}] = \frac{1}{2}\left(\frac{2}{3}D + D\right) + \frac{1}{2}T = \frac{5}{6}D + \frac{1}{2}T$$

and further $c_{\text{OPT}} = T$. Hence we get

$$c = \frac{\frac{5}{6}D + \frac{1}{2}T}{T} = \frac{1}{2} + \frac{5}{6} \cdot \frac{D}{T} \leq \frac{1}{2} + \frac{5}{6} \cdot \frac{D}{\frac{2}{3}D} = \frac{1}{2} + \frac{5}{4} = \frac{7}{4} < 2 \ .$$

- $T \geq D$: We have for the expected energy consumption of ALG

$$\mathbf{E}[C_{\text{ALG}}] = \frac{1}{2}\left(\frac{2}{3}D + D\right) + \frac{1}{2}(D + D) = \frac{5}{6}D + D = \frac{11}{6}D$$

and further $c_{\text{OPT}} = D$. Hence we get

$$c = \frac{\frac{11}{6}D}{D} = \frac{11}{6} < 2 \ .$$

Hence, the randomised algorithm is $\frac{11}{6}$-competitive which is better than any deterministic algorithm.

Note: This result, however, is not optimal yet. The best randomised algorithm uses a continuous probability distribution for the shutdown time and thereby achieves a competitive ratio of $e/(e-1) \approx 1.58$.

PhD-Scheduling [Exam]

a) (i) SMALLLOAD distributes the tasks as follows:

PhD student 1:	2	4	7
PhD student 2:	5	4	

OPT uses the following distribution (or another one with the same cost):

PhD student 1:	2	5	3
PhD student 2:	4	7	

SMALLLOAD thus distributes the tasks with cost $\text{ALG}(\sigma) = 13$ while OPT incurs a cost of $\text{OPT}(\sigma) = 11$. Hence,

$$\rho(\sigma) = \frac{\text{ALG}(\sigma)}{\text{OPT}(\sigma)} = \frac{13}{11} \ .$$

(ii) The following sequence results in a larger competitive ratio: $\sigma = 1, 1, 2$. We have $\text{ALG}(\sigma) = 3$ and $\text{OPT}(\sigma) = 2$ and thus

$$\rho(\sigma) = \frac{\text{ALG}(\sigma)}{\text{OPT}(\sigma)} = \frac{3}{2} \ .$$

(iii) See **b)**.

(iv) No, finding the optimal solution offline corresponds to solving the PARTITION-problem, which is NP-complete, thus presumably no efficient algorithm exists for the problem.

b) We first show a lower bound of $(2 - \frac{1}{m})$ on the competitive ratio of SMALLLOAD. To this end, we choose an input sequence that consists of $m(m-1)$ tasks of size 1 concluded with a task of size m, i.e. $\sigma = \underbrace{1, \ldots, 1}_{m(m-1)}, m$. After assigning the first $m(m-1)$ tasks, SMALLLOAD has assigned $m-1$ units to each of the m PhD students. The last task of size m incurs a load of $2m-1$ for the student to whom it is assigned.

The optimal algorithm assigns the first $m(m-1)$ taks to only $m-1$ students and the last (heavy) task to the remaining student. This results in a maximal load of m and we get the following lower bound for the competitive ratio:

$$c \geq \frac{\textsc{Alg}(\sigma)}{\textsc{Opt}(\sigma)} = \frac{2m-1}{m} = 2 - \frac{1}{m}$$

Now we shall show a matching upper bound for the competitive ratio. Let $\sigma = (e_1, e_2, \ldots)$ be an arbitrary input sequence. Without loss of generality, we assume s_1 to be the student with the maximal load for σ. Furthermore, let w be the effort of the last task T assigned s_1 and E the load of s_1 before assigning its last task. The load of all other students must be at least E since s_1 was the student with minimal load when he was assigned task T (otherwise another student would have received T). Hence, the sum of the loads of all students is at least $m \cdot E + w$ and hence

$$\textsc{Opt}(\sigma) \geq \frac{m \cdot E + w}{m} = E + \frac{w}{m} \ .$$

Using $\textsc{Opt}(\sigma) \geq w$, we get

$$\begin{aligned}
\textsc{Alg}(\sigma) &= w + E \\
&\leq w + \textsc{Opt}(\sigma) - \frac{w}{m} \\
&= \textsc{Opt}(\sigma) + \left(1 - \frac{1}{m}\right) w \\
&\leq \textsc{Opt}(\sigma) + \left(1 - \frac{1}{m}\right) \textsc{Opt}(\sigma) \\
&= \left(2 - \frac{1}{m}\right) \textsc{Opt}(\sigma)
\end{aligned}$$