Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Distributed Computing

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Discrete Event Systems

Solution to Exercise Sheet 8

1 Dolce Vita in Rome

We define the following random variables.

$$X = \text{number of ice creams bought in total}$$
$$X_i = \text{indicator variable for buying ice cream at shop } i$$

That means X_i is 1 if Hector and Rachel buy ice cream at shop i and 0 otherwise. Since the probability that the i-th shop is the best so far equals $\frac{1}{i}$ and the expectation of an indicator variable is simply the probability of it being 1, we have

$$\mathbf{E}[X_i] = \frac{1}{i} \ .$$

Furthermore, we can express X as $\sum_{i=1}^{n} X_i$ and by using linearity of expectation, we obtain:

$$\mathbf{E}[X] = \mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} \frac{1}{i} = H_n \ .$$

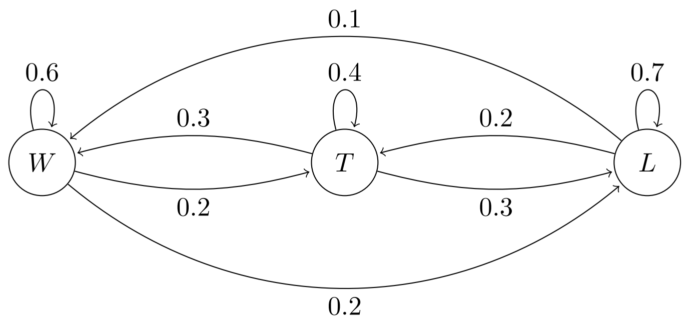
Here H_n is the n-th *Harmonic Number*. H_n grows about as fast as the natural logarithm of n. The reason for this is that the sum of the first n harmonic numbers can be approximated by

$$\int_1^n \frac{1}{x} \mathrm{d}x = \ln(n) \ .$$

More precisely, we have $H_n = \ln(n) + \mathcal{O}(1)$ and thus the two students roughly consume a logarithmic number of ice creams (in the total number of shops n).

2 Soccer Betting

a) The following Markov chain models the different transition probabilities (W:Win, T:Tie, L:Loss):



b) The transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} .$$

As you might have noticed, we gave redundant information here. You only need the information that the FCB lost its last game. Thus, the Markov chain is currently in the state L and hence, the initial vector is $q_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. The probability distribution q_2 for the game against the FC Zurich is therefore given by

$$\begin{aligned} q_2 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P &= \begin{pmatrix} 0.1 & 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \\ &= \begin{pmatrix} 0.19 & 0.24 & 0.57 \end{pmatrix} . \end{aligned}$$

(Note that q_0 must be a row vector, not a column vector.)

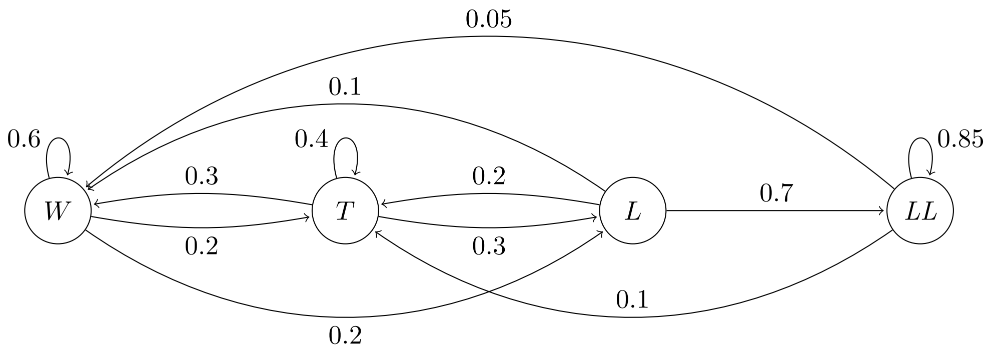
Hint: We exploited the associativity of the matrix multiplication to avoid having to calculate P^2 explicitly. This is usually a good “trick” to avoid extensive and error-prone calculations if no calculator is at hand (as for example in an exam situation ☺).

Given the quotas of the exercise, the expected return for each of the three possibilities (W, T, L) calculates as follows.

$$\begin{aligned} \mathbf{E}[W] &= 0.19 \cdot 3.5 = 0.665 \\ \mathbf{E}[T] &= 0.24 \cdot 4 \quad = 0.96 \\ \mathbf{E}[L] &= 0.57 \cdot 1.5 = 0.855 \end{aligned}$$

Therefore, the best choice is not to bet at all since the expected return is smaller than 1 for every choice. If a “sales representative” of the Swiss gambling mafia were to force you to bet, you would be best off with betting on a tie, though.

c) The new Markov chain model looks like this. In addition to the three states W, T, and L, there is now a new state LL which is reached if the team has lost twice in a row.



The new transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} . \qquad (1)$$

As the FCB has and lost its last two games, the Markov chain is currently in the state $q_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. The probabilities for the game against the FC Zurich can again be computed as follows.

$$q_3 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P = \begin{pmatrix} 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1025 & 0.135 & 0.04 & 0.7225 \end{pmatrix}$$

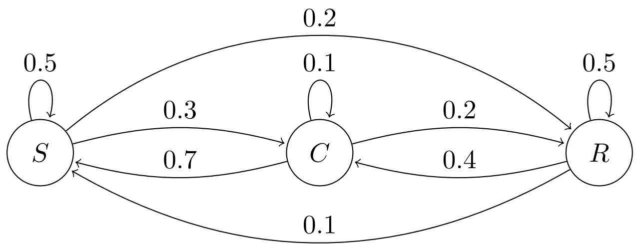
Finally, we can compute the expected profit for each of the three possible bets:

$$\begin{aligned} \mathbf{E}[W] &= 0.1025 \cdot 3.5 &&= 0.35875 \\ \mathbf{E}[T] &= 0.135 \cdot 4 &&= 0.54 \\ \mathbf{E}[L] &= (0.04 + 0.7225) \cdot 1.5 &&= 1.14375 \ . \end{aligned}$$

Now, the best choice is to bet on a loss. Clearly, the addition of the state LL worsens the situation for FCB.

3 The Winter Coat Problem

a) The following Markov chain models the weather situation of Robinson's island.



b) We need to determine the expected hitting time h_{SS}. Using the formula of slide 35, we obtain the following equation system:

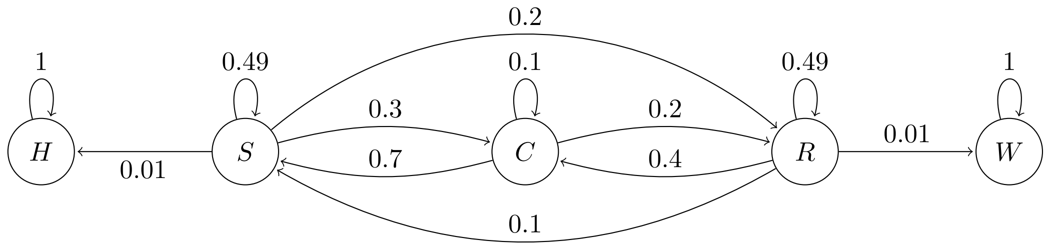
$$h_{SS} = 1 + 0.3h_{CS} + 0.2h_{RS} \tag{2}$$
$$h_{CS} = 1 + 0.1h_{CS} + 0.2h_{RS} \tag{3}$$
$$h_{RS} = 1 + 0.4h_{CS} + 0.5h_{RS} \tag{4}$$

(2) and (3) yield that $h_{CS} = \frac{5}{6}h_{SS}$, from (2) and (4) we obtain that $h_{RS} = \frac{40}{23}h_{SS} - \frac{10}{23}$. Plugging these results into (2), we obtain

$$h_{SS} = 1 + 0.3\left(\frac{5}{6}h_{SS}\right) + 0.2\left(\frac{40}{23}h_{SS} - \frac{10}{23}\right)$$
$$\Leftrightarrow \quad h_{SS} = \frac{1 - \frac{2}{23}}{1 - \frac{1}{4} - \frac{8}{23}} = \frac{84}{37} \approx 2.27$$

Thus, Mr. Robinson has to wait 2.27 days (in expectation) until having again a sunny day. (*Note:* Between two sunny days, there are (in expectation) 1.27 non-sunny days.)

c) The modified Markov chain looks as follows:



d) We need to determine the arrival probability f_{SW}, the probability that the weather will turn to winter. Using the formula of slide 35, we obtain the following equation system:

$$f_{SW} = 0 + 0.3f_{CW} + 0.2f_{RW} + 0.49f_{SW} + 0.01f_{HW} \quad (5)$$
$$f_{CW} = 0 + 0.7f_{SW} + 0.2f_{RW} + 0.1f_{CW} \quad (6)$$
$$f_{RW} = 0.01 + 0.4f_{CW} + 0.1f_{SW} + 0.49f_{RW} \quad (7)$$
$$f_{HW} = 0 \quad (8)$$

Solving the equation system yields

$$f_{SW} = \frac{240}{619}, \; f_{RW} = \frac{249}{619}, \; f_{CW} = \frac{242}{619}$$

And therefore, the probability that the weather turns to winter (snowing) and Mr. Robinson needs a winter coat is $\frac{240}{619} \approx 0.39$. Note that $f_{SH} = 1 - f_{SW} = \frac{379}{619}$ because all state sequences that do not end up in state W eventually end up in state H.