Chapter 5

Worst-Case Event Systems

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Overview: Worst-Case Analysis of DES

- Ski Rental
  - Randomized Ski Rental
  - Lower Bounds
- The TCP Acknowledgement Problem
- The TCP Congestion Control Problem
  - Bandwidth in a Fixed Interval
  - Multiplicatively Changing Bandwidth
  - Changes with Bursts
- Many application domains are not Poisson distributed!
  - sometimes it makes sense to assume that events are distributed in the worst possible way (e.g. in networks, packets often arrive in bursts)

Theory of Renting Skis

- Scenario
  - you start a new hobby, e.g. skiing
  - you don't know whether you will like it
  - expensive equipment : ≈1 kFr
- 3 Alternatives
  - just buy a new equipment (optimistic)
  - always renting (pessimistic)
  - first rent it a few times before you buy (down-to-earth)
- You choose the pragmatic way, but Murphy's law will strike!
  - first you rent, but as soon as you buy, you will lose interest in skiing

Ski Rental Problem

- Expenses
  - buying: 1 kFr
  - renting: 1 kFr per month
- Scenario
  - first rent it for z months, then buy it
  - after u months you will lose your interest in skiing

  2 cases:

  $u \le z \rightarrow cost_z(u) = u$ kFr

  $u > z \rightarrow cost_z(u) = (z + 1)$ kFr

- If you are a clairvoyant, then …

  $u \le 1$ month → just renting is better → $cost_{opt}(u) = u$ kFr

  $u > 1$ month → just buying is better → $cost_{opt}(u) = 1$ kFr

  → $cost_{opt}(u) = \min(u, 1)$

Competitive Analysis

- Definition
  An online algorithm A is *c*-competitive if for all finite input sequences *I*
  $$cost_A(I) \le c\ cost_{opt}(I) + k$$
  where *k* is a constant independent of the input.
  If *k* = 0, then the online algorithm is called strictly *c*-competitive.

- When strictly *c*-competitive, it holds
  $$\frac{cost_A(u)}{cost_{opt}(u)} \le c$$

- Example
  - Ski rental is strictly 2-competive. The best algorithm is *z* = 1.

Randomized Ski Rental

- Deterministic Algorithm
  - has a big handicap, because the adversary knows *z* and can always present a *u* which is worst-case for the algorithm
  - only hope: algorithm makes random decisions
- Randomized Algorithm
  - chooses randomly between 2 values z_1 und z_2 (with $z_1 < z_2$) with probabilities p_1 and $p_2 = (1 - p_1)$

$$cost_A(u) = \begin{cases} u & \text{if } u \le z_1 \\ p_1 \cdot (z_1 + 1) + p_2 \cdot u & \text{if } z_1 < u \le z_2 \\ p_1 \cdot (z_1 + 1) + p_2 \cdot (z_2 + 1) & \text{if } z_2 < u \end{cases}$$

  - adversary chooses randomly
    - $u_1 = z_1 + \varepsilon$ with probability q_1
    - $u_2 = z_2 + \varepsilon$ with probability $q_2 = 1 - q_1$
- Example
  - $z_1 = ½, z_2 = 1, p_1 = 2/5, p_2 = 3/5$
  - $E[c] = cost_A / cost_{opt} = 1.8$

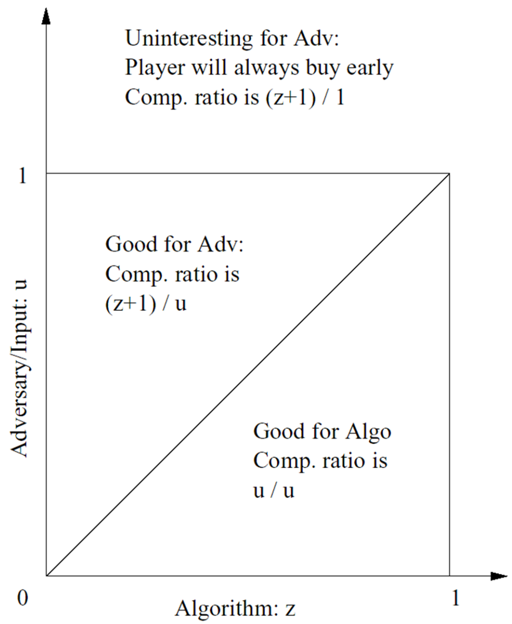
What about choosing randomly between more than 2 values???

Randomized Ski Rental with infinitely many Values (1)

- Let *r*(*u*, *z*) be the competitive ratio for all pairs of *u* and *z*
- We are looking for the expected competitive ratio E[c]

- Adversary chooses *u* with uniform distribution

$$E[c] = \frac{\iint r(u,z)dzdu}{\iint dzdu}$$
$$= \frac{1}{2} + \int_{u=0}^{1}\int_{z=0}^{u} \frac{z+1}{u} dzdu$$
$$= 1.75$$



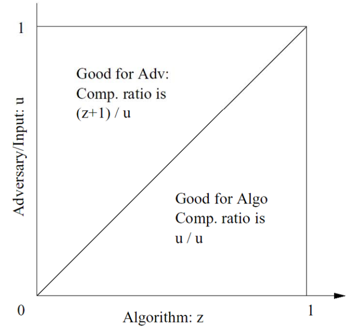
Randomized Ski Rental with infinitely many Values (2)

- Algorithm chooses *z* with probability distribution *p*(*z*)
  - it chooses *p*(*z*) such that it minimizes E[c]
- Adversary chooses *u* with probability distribution *d*(*u*)
  - it chooses *d*(*u*) such that it maximized E[c]

$$E[c] = \frac{\int_0^1 \int_0^u (z+1)p(z)d(u)dzdu + \int_0^1 \int_u^1 up(z)d(u)dzdu}{\int_0^1 \int_0^1 up(z)d(u)dzdu}$$

$$\int p(z) = \int d(u) = 1$$

- How to find these probability distributions?
  - This is a very hard task!
  - → We should make the problem independent of the adversarial distribution *d*(*u*).



Randomized Ski Rental with infinitely many Values (3)

- Idea

  Choose the algorithm's probability function $p(z)$ such that
  $\text{cost}_A(u) \le c\ \text{cost}_{opt}(u)$ for all u
  → adversarial distribution $d(u)$ doesn't matter anymore

- $\text{cost}_{opt}(u) = u$ for all u between 0 und 1

$$\int_0^u (z+1)p(z)dz + \int_u^1 u \cdot p(z)dz \le c \cdot u$$

$$\text{with} \int_0^1 p(z)dz = 1$$

- Having a hunch: the best probability function $p(z)$ will be an equality
  → With $p(z) = \frac{e^z}{e-1}$ we have an algorithm that is $\frac{e}{e-1}$-competitive in expectation.

Can we get any better??? → Lower Bounds

- Von Neumann / Yao Principle

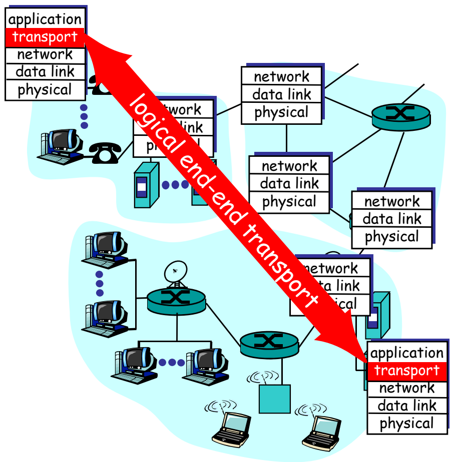
  Choose a distribution over problem instances (for ski rental, e.g. $d(u)$).
  If for this distribution all deterministic algorithms cost at least c,
  then c is a lower bound for the best possible randomized algorithm.

- Ski Rental
  - we are in a lucky situation, because we can parameterize all possible deterministic algorithms by $z \ge 0$
  - choose a distribution of inputs with $d(u) \ge 0$ and $\int d(u) = 1$

- Examples: $d(u) = ½$ for $0 \le u \le 1$ and $d(\infty) = ½$
  - → $\text{cost}_{z=0}(d(u)) = 1$ | $\text{cost}_{z\le 1}(d(u)) = 1 + z/2 - z^2/4 \ge 1$
  - → $\text{cost}_{z=1}(d(u)) = 5/4$ | $\text{cost}_{z>1}(d(u)) = ¼ + (z+1)/2 > 5/4$
  - → $\text{cost}_{opt}(d(u)) = ¾$
  - → $c\ /\ \text{cost}_{opt} = 1/¾ = 4/3 = 1.33$

TCP: Transmission Control Protocol

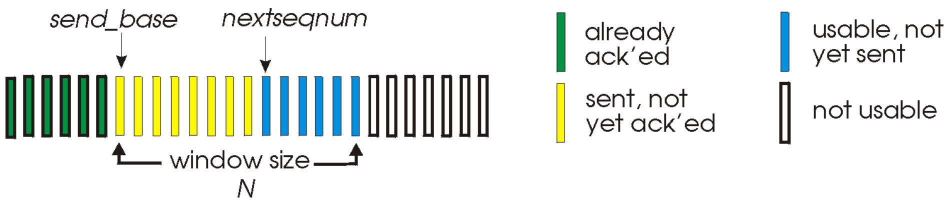
- Layer 4 Networking Protocol
  - transmission error handling
  - correct ordering of packets
  - exponential ("friendly") slow start mechanism: should prevent network overloading by new connections
  - flow control: prevents buffer overloading
  - congestion control: should prevent network overloading



Packet Acknowledgment

Sender

- Sequence number in packet header
- "Window" of up to N consecutive unack'ed packets allowed



- ACK(n): ACKs all packets up to and including sequence number n
  - a.k.a. cumulative ACK
  - sender may get duplicate ACKs
- timer for each in-flight packet
- *timeout(n):* retransmit packet n and all higher seq# packets in window

The TCP Acknowledgment Problem

- Definition

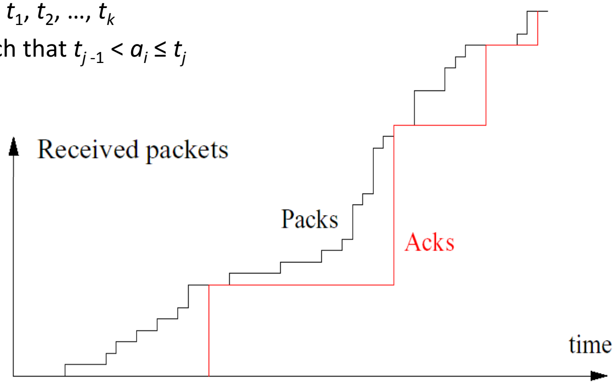
  The receiver's goal is a scheme which minimizes the number of acknowledgments plus the sum of the latencies for each packet, where the latency of a packet is the time difference from arrival to acknowledgment.
- Given

  n packet arrivals, at times: $a_1, a_2, \ldots, a_n$

  k acknowledgments, at times $t_1, t_2, \ldots, t_k$

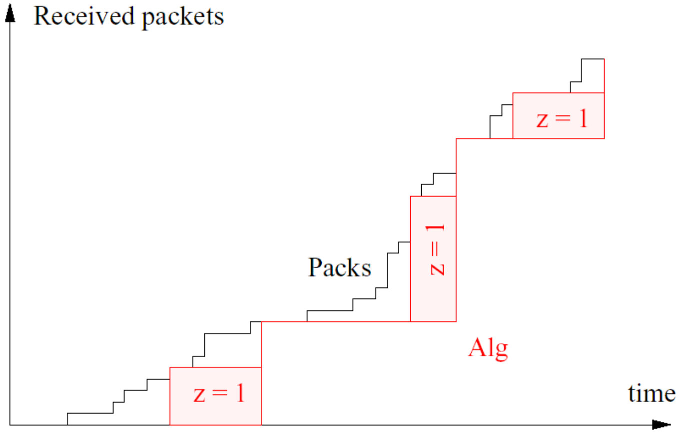
  latency(i) = $t_j - a_i$, where j such that $t_{j-1} < a_i \le t_j$
- Minimize

$$\left(k + \sum_{i=1}^{n} \text{latency}(i)\right)$$



The TCP Acknowledgment Problem: z=1 Algorithm (1)

- $z = 1$ Algorithm is: Whenever a rectangle with area $z = 1$ does fit between the two curves, the receiver sends an acknowledgement, acknowledging all previous packets.

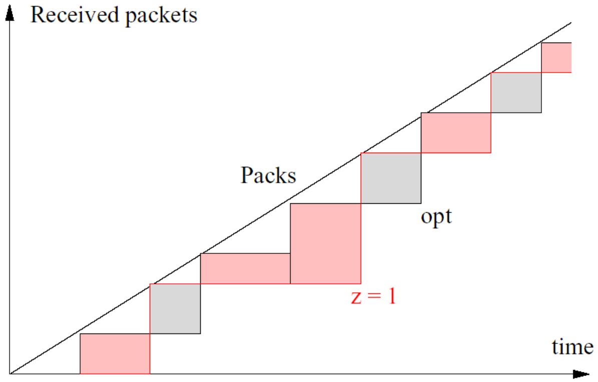


The TCP Acknowledgment Problem: z=1 Algorithm (2)

- Lemma
  - The optimal algorithm sends an ACK between any pair of consecutive ACKs by algorithm with $z = 1$.
- Proof
  - For the sake of contradiction, assume that, among all algorithms who achieve the minimum possible cost, *there is no algorithm* which sends an ACK between two ACKs of the $z = 1$ algorithm.
  - We propose to send an additional ACK at the beginning (left side) of each $z = 1$ rectangle.
    Since this ACK saves latency 1, it compensates the cost of the extra ACK.
  - That is, there is an optimal algorithm who chooses this extra ACK.

The TCP Acknowledgment Problem: z=1 Algorithm (3)

- Theorem: The $z = 1$ algorithm is 2-competitive.

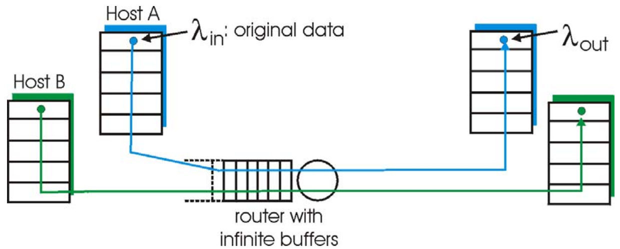


- Similarity to Ski Rental
  - it's possible to choose any z
  - if you wait for a rectangle of size z with probability $p(z) = e^z/(e-1)$
    → randomized TCP ACK solution, which is $e/(e-1)$ competitive

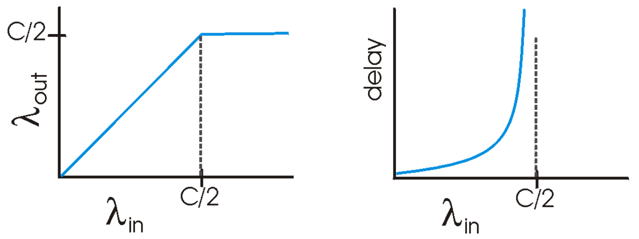
Simple TCP Congestion Scenario

congestion
too many sources sending too much data too fast for the *network* to handle

- two equal senders, two receivers
- one router with infinite buffer space and with service rate C



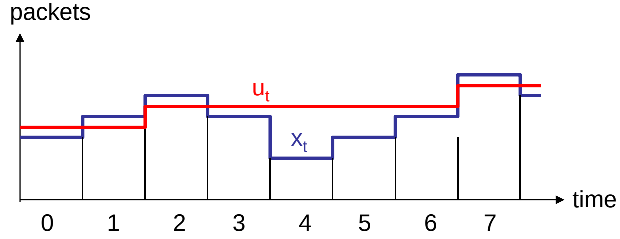
- large delays when congested
- maximum achievable throughput



The TCP Congestion Control Problem

- Main Question

  How many packets per second can a sender inject into the network without overloading it?
- Assumptions
  - sender does not know the bandwidth between itself and the receiver
  - the bandwidth might change over time
- Model
  - time divided into periods { t }
  - unknown bandwidth threshold u_t
  - sender transmits x_t packets



- Severe Cost and Gain Function
  - $gain_t = u_t - cost_t$
  - $x_t \leq u_t$: $cost_t = u_t - x_t$ → $gain_t = x_t$
  - $x_t > u_t$: $cost_t = u_t$ → $gain_t = 0$

The TCP Congestion Control Problem: The Dynamic Model

- Competitive Analysis Definition

  An online algorithm A is strictly c-competitive if for all finite input sequences *I*

  $$cost_A(I) \leq c \cdot cost_{opt}(I)$$

  or

  $$c \cdot gain_A(I) \geq gain_{opt}(I).$$

- The Dynamic Model
  - algorithm: chooses a sequence { x_t }
  - adversary: knows the algorithm's sequence and chooses a sequence { u_t }
- Problem
  - Adversary is too strong: $\forall t$: $u_t < x_t$ → $gain_A = 0$
- Reasonable restrictions
  - Bandwidth in a fixed interval: $u_t \in [a, b]$
  - Multiplicatively or additively changing bandwidth from step to step
  - Changes with bursts

Bandwidth in a Fixed Interval: Deterministic Algorithm

- Preconditions
  - adversary chooses $u_t \in [a, b]$
  - algorithm is aware of the lower bound a and the upper bound b
- Deterministic Algorithm
  - If the algorithm plays $x_t > a$ in round t, then the adversary plays $u_t = a$ → gain = 0
  - Therefore the algorithm must play $x_t = a$ in each round in order to have at least gain = a.
  - The adversary knows this, and will therefore play $u_t = b$.
  - Therefore, $gain_{Alg} = a$, $gain_{opt} = b$, competitive ratio $c = b/a$.

Bandwidth in a Fixed Interval: Randomized Algorithm

- Let's try the ski rental trick!
  - For all possible inputs $u \in [a, b]$ we want the same competitive ratio:

    $$c \; gain_{Alg}(u) = gain_{opt}(u) = u$$

- Randomized Algorithm
  - We choose $x = a$ with probability p_a, and any value in $x \in (a, b]$ with probability density function $p(x)$, with $p_a + \int_a^b p(x)dx = 1.$

- Theorems
  - There is an algorithm that is c-competitive, with $c = 1 + \ln(b/a)$.
  - There is no randomized algorithm which is better than c-competitive, with $c = 1 + \ln(b/a)$.

- Remark
  - Upper and lower bound are tight.

Multiplicatively Changing Bandwidth

- Preconditions
  - adversary chooses u_{t+1} such that $u_t/\mu \leq u_{t+1} \leq \mu\, u_t$, with $\mu \geq 1$, e.g. 1.05
  - algorithm knows u_1 and μ

- Algorithm A_1
  - after a successful transmission in period t, the algorithm chooses $x_{t+1} = \mu\, x_t$
  - otherwise: $x_{t+1} = x_t/\mu^3$
- Theorem
  - The algorithm A_1 is $(\mu^4 + \mu)$-competitive

- Algorithm A_2
  - after a successful transmission in period t, the algorithm chooses $x_{t+1} = \mu\, x_t$
  - otherwise: $x_{t+1} = x_t/2$
- Theorem
  - The algorithm A_2 is (4μ)-competitive

Changes with Bursts

- Bursty Adversary
  - 2 parameters: $\mu \geq 1$ and maximum burst factor $B \geq 1$
  - adversary chooses u_{t+1} from the interval $[\frac{u_t}{\beta_t \mu}, u_t \cdot \beta_t \cdot \mu]$ where $\beta_t = \min\{B, \beta_{t-1}\frac{\mu}{c_{t-1}}\}$ is the burst factor at time t and where $c_{t-1} = u_t/u_{t-1}$ if $u_t > u_{t-1}$ and u_{t-1}/u_t otherwise