Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Distributed Computing

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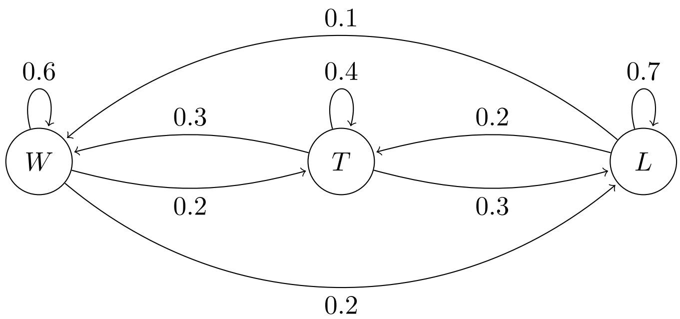
Discrete Event Systems

Solution to Exercise Sheet 6

1 Soccer Betting

a) The following Markov chain models the different transition probabilities (W:Win, T:Tie, L:Loss):



b) The transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \quad .$$

As you might have noticed, we gave redundant information here. You only need the information that the FCB lost its last game. Thus, the Markov chain is currently in the state L and hence, the initial vector is $q_0 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$. The probability distribution q_2 for the game against the FC Zurich is therefore given by

$$\begin{aligned} q_2 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P &= \begin{pmatrix} 0.1 & 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \\ &= \begin{pmatrix} 0.19 & 0.24 & 0.57 \end{pmatrix} \quad . \end{aligned}$$

(Note that q_0 must be a row vector, not a column vector.)

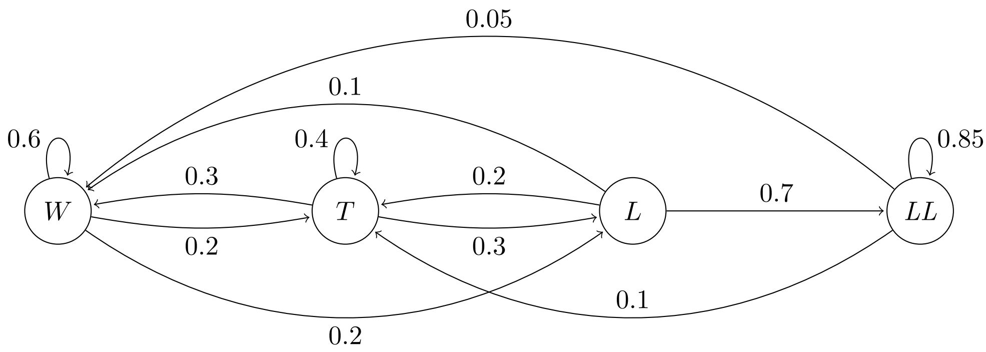
Hint: We exploited the associativity of the matrix multiplication to avoid having to calculate P^2 explicitly. This is usually a good "trick" to avoid extensive and error-prone calculations if no calculator is at hand (as for example in an exam situation ☺).

Given the quotas of the exercise, the expected return for each of the three possibilities (W, T, L) calculates as follows.

$$\begin{aligned} \mathbf{E}[W] &= 0.19 \cdot 3.5 = 0.665 \\ \mathbf{E}[T] &= 0.24 \cdot 4 \quad = 0.96 \\ \mathbf{E}[L] &= 0.57 \cdot 1.5 = 0.855 \end{aligned}$$

Therefore, the best choice is not to bet at all since the expected return is smaller than 1 for every choice. If a "sales representative" of the Swiss gambling mafia were to force you to bet, you would be best off with betting on a tie, though.

c) The new Markov chain model looks like this. In addition to the three states W, T, and L, there is now a new state LL which is reached if the team has lost twice in a row.



The new transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} . \tag{1}$$

As the FCB has and lost its last two games, the Markov chain is currently in the state $q_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$. The probabilities for the game against the FC Zurich can again be computed as follows.

$$q_3 = q_0 \cdot P^2 = (q_0 \cdot P) \cdot P = \begin{pmatrix} 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} \cdot \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix}$$

$$= \begin{pmatrix} 0.1025 & 0.135 & 0.04 & 0.7225 \end{pmatrix}$$

Finally, we can compute the expected profit for each of the three possible bets:

$$\begin{aligned} \mathbf{E}[W] &= 0.1025 \cdot 3.5 &&= 0.35875 \\ \mathbf{E}[T] &= 0.135 \cdot 4 &&= 0.54 \\ \mathbf{E}[L] &= (0.04 + 0.7225) \cdot 1.5 &&= 1.14375 \ . \end{aligned}$$

Now, the best choice is to bet on a loss. Clearly, the addition of the state LL worsens the situation for FCB.

2 Probability of Arrival

The proof is similar to the one about the expected hitting time h_{ij} (see script). We express f_{ij} as a condition probability that depends on the result of the first step in the Markov chain. Recall that the random variable T_{ij} is the *hitting time*, that is, the number of steps from i to j. We get $Pr[T_{ij} < \infty \mid X_1 = k] = \Pr[T_{kj} < \infty] = f_{kj}$ for $k \neq j$ and $Pr[T_{ij} < \infty \mid X_1 = j] = 1$. We can therefore write f_{ij} as follows.

$$\begin{aligned}
f_{ij} = \Pr[T_{ij} < \infty] &= \sum_{k \in S} \Pr[T_{ij} < \infty \mid X_1 = k] \cdot p_{ik} \\
&= p_{ij} \cdot \Pr[T_{ij} < \infty \mid X_1 = j] + \sum_{k \neq j} \Pr[T_{ij} < \infty \mid X_1 = k] \cdot p_{ik} \\
&= p_{ij} + \sum_{k \neq j} p_{ik} f_{kj}
\end{aligned}$$

3 Basketball

a) This exercise is a good example to illustrate that most exercises allow several differing solutions.

Variant A. Let X be a random variable for the number of shots scored by Mario and X_i an indicator variable that the i-th shot scores. Then obviously $X = \sum_{i=1}^{n} X_i$ when n is the number of shots performed. The probability that the i-th attempt scores is p as given in the exercise. Hence, we can use linearity of expectation to obtain the expectation of X.

$$\mathbf{E}[X] = \mathbf{E}\Big[\sum_{i=1}^{n} X_i\Big] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} p = n \cdot p$$

We want Mario to score m times.

$$\mathbf{E}[X] = n \cdot p = m \quad \Longleftrightarrow \quad n = \frac{m}{p}$$

Hence, Mario needs $\frac{m}{p}$ attempts to score m times. After these $\frac{m}{p}$ attempts, Mario has scored an expected m hits and he has missed expected $\frac{m}{p} - m$ times. Hence, he does an expected $10(\frac{m}{p} - m)$ push-ups in the game.

Variant B. We define a random variable X that counts the number of attempts until we miss for the first time. X is distributed as follows:

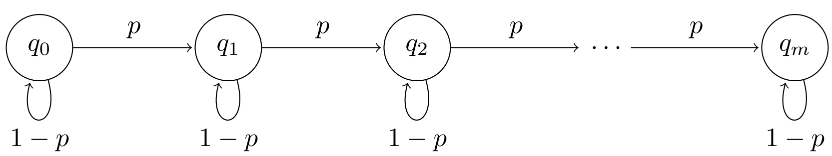
$$\begin{aligned}
\Pr[X = 1] &= (1 - p) \\
\Pr[X = 2] &= p(1 - p) \\
&\vdots \\
\Pr[X = i] &= p^{i-1}(1 - p)
\end{aligned}$$

We say that X is geometrically distributed with parameter $(1-p)$ or write $X \sim \text{Geom}(1-p)$. The expected value of a geometrically distributed random variable with parameter α is $\frac{1}{\alpha}$.

$$\mathbf{E}[X] = \frac{1}{1 - p} \quad .$$

Again, due to the linearity of the expected value we may think of the game as Mario scoring $\mathbf{E}[X] - 1$ hits, missing once, scoring the next $\mathbf{E}[X] - 1$ hits, missing again, and so forth until he scored a total of m hits. The question of how often Mario misses now translates to the question of how many series of $\mathbf{E}[X] - 1$ successful attempts he needs in order to score m times, and we get $10 \cdot \frac{m}{\mathbf{E}[X]-1} = 10 \cdot (\frac{m}{p} - m)$ push-ups in expectation.

Variant C (Markov Chain). The following Markov chain models Mario's game.



In a state q_i Mario has scored i hits. To learn the expected number of attempts until Mario has scored m hits we can simply compute the hitting time h_{0m} from q_0 to q_m.

$$\begin{aligned}
h_{0m} &= 1 + \sum_{k \neq m} p_{0k} h_{km} = 1 + p_{00} h_{0m} + p_{01} h_{1m} \\
h_{0m} &= \frac{1 + p_{01} h_{1m}}{1 - p_{00}} = \frac{1 + p h_{1m}}{p} = \frac{1}{p} + h_{1m} \\
h_{1m} &= 1 + p_{11} h_{1m} + p_{12} h_{2m} \iff h_{1m} = \frac{1}{p} + h_{2m} \\
h_{0m} &= \frac{1}{p} + h_{1m} = \frac{2}{p} + h_{2m} = \ldots = \frac{m}{p} + h_{mm} = \frac{m}{p}
\end{aligned}$$

By subtracting the m successful attempts, we get an expected $\frac{m}{p} - m$ misses and hence Mario does $10(\frac{m}{p} - m)$ push-ups in expectation.

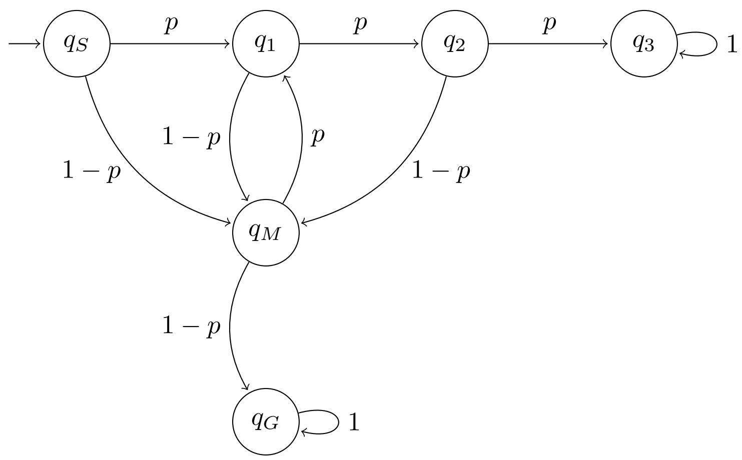
b) Each sequence of (at most m) throws where Luigi tries to score m times is called a *round*. A non-successful round is followed by push-ups.

Let X be a random variable for the number of rounds that Luigi has to perform until he hits m shots straight. The probability that Luigi scores m consecutive shots is p^m. Observe that X is geometrically distributed with parameter p^m (cf. Exercise 3a variant B) and hence

$$\mathbf{E}[X] = \frac{1}{p^m} \quad .$$

In the last round (which was successful), Luigi does not do any push-ups, hence we expect him to do $10 \cdot \left(\frac{1}{p^m} - 1\right)$ push-ups.

c) The following Markov chain models Trudy's game.



In state q_i Trudy has scored i hits in a row, in q_M she has missed once, in q_G she has missed twice in a row and gives up.

(i) We determine the probability f_{S3} of reaching the accepting state q_3 from the start state q_S.

$$\begin{aligned}
f_{S3} &= p \cdot f_{13} + (1-p) \cdot f_{M3} \\
f_{13} &= p \cdot f_{23} + (1-p) \cdot f_{M3} \\
f_{23} &= p + (1-p) \cdot f_{M3} \\
f_{M3} &= p \cdot f_{13}
\end{aligned}$$

$$
\begin{aligned}
f_{13} &= p^2 + (1-p)p^2 \cdot f_{13} + (1-p)p \cdot f_{13} \\
&= \frac{p^2}{1+p^3-p} = 0.4
\end{aligned}
$$

$$
\begin{aligned}
f_{S3} &= p \cdot \frac{p^2}{1+p^3-p} + (1-p)p \cdot \frac{p^2}{1+p^3-p} \\
&= \frac{2p^3-p^4}{1+p^3-p} \\
&= 0.3
\end{aligned}
$$

The probability that Trudy scores 3 times in a row is 0.3. The probability that she gives up is 0.7. This is because q_3 and q_G are the only absorbing states, i.e., all other states have probability mass of 0 in the steady state.

(ii) To get the number of push-ups we define a random variable Z that counts how often the system passes state q_M before either ending up in state q_3 or in state q_G. E.g., the probability $P[Z=1]$ of passing q_M exactly once equals the probability of getting from q_S to q_M without being absorbed by q_3 and then ending up directly in q_G or q_3, i.e. $\Pr[Z=1] = P_{SM} \cdot (P_{MG} + P_{M3})$ where P_{ij} is the probability of getting from q_i to q_j without passing q_M on the way. Z has the following probability distribution:

$$
\begin{aligned}
\Pr[Z=0] &= 1 - P_{SM} \\
\Pr[Z=1] &= P_{SM} \cdot (P_{MG} + P_{M3}) \\
\Pr[Z=2] &= P_{SM} \cdot P_{MM} \cdot (P_{MG} + P_{M3}) \\
\Pr[Z=3] &= P_{SM} \cdot P_{MM}^2 \cdot (P_{MG} + P_{M3}) \\
&\vdots \\
\Pr[Z=i] &= P_{SM} \cdot P_{MM}^{i-1} \cdot (P_{MG} + P_{M3})
\end{aligned}
$$

The probability of passing q_M exactly i times equals the probability of getting from q_S to q_M and from q_M to q_M again $i-1$ times and then ending up directly in q_G or q_3. As the Markov chain is not too complicated we can compute the needed P_{ij} rather easily and get $P_{SM} = 1-p^3$, $P_{MM} = p - p^3$, $P_{MG} = 1-p$, and $P_{M3} = p^3$. The expected number of misses is

$$
\begin{aligned}
\mathbf{E}[Z] &= \sum_{i=1}^{\infty} i \cdot \Pr[Z=i] \\
&= \sum_{i=1}^{\infty} i \cdot P_{SM} \cdot P_{MM}^{i-1} \cdot (P_{MG} + P_{M3}) \\
&= P_{SM} \cdot (P_{MG} + P_{M3}) \cdot \sum_{i=1}^{\infty} i \cdot P_{MM}^{i-1} \\
&= \frac{P_{SM} \cdot (P_{MG} + P_{M3})}{(1-P_{MM})^2} \\
&= \frac{(1-p^3) \cdot (1-p+p^3)}{(1-p+p^3)^2} = \frac{1-p^3}{1-p+p^3} \\
&= \frac{1-\frac{1}{8}}{1-\frac{1}{2}+\frac{1}{8}} = \frac{7}{5} = 1.4.
\end{aligned}
$$

Hence, Trudy does 14 push-ups in expectation.

Variant. We already know that Trudy gives up with a probability 0.7. Each time Trudy is in q_M she gets to q_G with probability $1 - p$. Hence it must hold that $\mathbf{E}[Z] \cdot (1-p) = 0.7$. This yields for the expected number of push-ups

$$10 \cdot \mathbf{E}[Z] = 10 \cdot \frac{0.7}{1-p} = 10 \cdot 2 \cdot 0.7 = 14.$$