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Discrete Event Systems

Exercise Sheet 6

1 Soccer Betting

The *FC Basel* soccer club is a particularly moody team. Upon winning a game, they tend to win subsequent games. After losing a game, however, they often end up losing the next game as well. A group of international scientists, consisting of soccer experts, mathematicians, and psychologists, has recently conducted a thorough analysis of this behavior. In particular, they have discovered that upon winning a game, the FCB wins the next game with a probability of 0.6 as well. With probabilities 0.2 each, the next game will be a tie or a loss. After a loss, the FCB will win/tie/lose its next game with probability 0.1/0.2/0.7, respectively. Finally, after a tie, the next game being a win or a loss is equally probable. The probability that the next game also ends up being a tie is 0.4.

a) Model the FCB's moodiness using a Markov chain.

b) In *two games* from now (they will play one game in between), the FCB will play against the FC Zurich. The Swiss TOTO offers you the following odds:

Win: 3.5 Tie: 4.0 Loss: 1.5

Given that the FCB won the game three games ago, but lost the last two games, what would be your bet? Why?

c) More recent studies have shown that the FCB is even moodier than expected. In fact, after losing *two games in a row*, the probability of winning its next game reduces to 0.05, that of getting a tie to 0.1. Change your Markov chain model to incorporate the new findings. How does the change influence your bet?

2 Night Watch

In order to improve their poor financial situation, Jochen and Georg also work at nights. Their task is to guard a famous Swiss bank which, from an architectonic perspective, looks as follows:

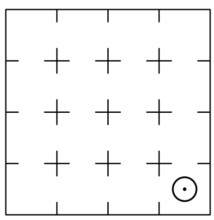


Figure 1: Offices of a Swiss bank. There are 4x4 rooms, all connected by doors as indicated.

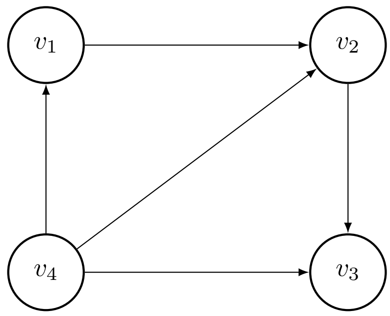
In a first scenario, Jochen and Georg always stay together. They start in the room on the upper left. Every minute, they change to the next room, which is chosen uniformly at random from all possible (adjacent) rooms.

a) Compute the probability (in the steady state) that Jochen and Georg are in the room where the thief enters the bank (indicated with $\odot$)!

b) Since Jochen and Georg are very strong, they can easily catch a thief on their own. Thus, in a second scenario, they decide that it would be smarter to patrol individually (but they still start in the same room): After every minute, each of them chooses the next room *independently*. What is now the probability that at least one of them is in the room where the thief enters?

3 PageRank

Usually, one would compute the PageRank of a network not by hand, but by running an algorithm on a computer – since the relevant instances are far too tedious to compute by hand. Therefore we only consider a very small example with four nodes for this exercise:



a) The naïve suggestion is to simply use the nodes' (in-)degree as a rank, by computing $(1,1,1,1)\cdot A$, where A is the adjacency matrix of the graph. What are the naïve PageRanks, and why might that not be a good idea?

b) An improved-naïve version would normalize each non-zero row in the adjacency matrix by dividing each row by the sum of its entries. Denote the obtained matrix by A'. What are the improved-naïve PageRanks $(1,1,1,1)\cdot A'$?

c) Since websites with higher rank should have a larger impact on the rank of the nodes they link to, you decide to iterate the process from **b)**. Denoting $(1,1,1,1)$ with q_0, calculate the iterated version of this "PageRank" by computing

$$\begin{aligned} q_1 &= q_0 \cdot A' \\ q_2 &= q_1 \cdot A' \\ q_3 &= q_2 \cdot A' \\ &\vdots \end{aligned}$$

(You should only need to iterate very few times.) When does the rank converge, and what is the issue? Would the same happen when using the Google matrix?

4 Probability of Arrival

In the script, there is a lemma saying that the probability of arrival in a markov chain can be computed using the formula

$$f_{ij} = p_{ij} + \sum_{k:k\neq j} p_{ik} f_{kj} \quad .$$

Prove this lemma.

5 Basketball [Exam]

Mario, Luigi and Trudy meet to play basketball. To improve their scoring abilities, Mario suggests the following game: Each player has to score m times. After each miss, he has to perform 10 push-ups.

a) Assume that Mario always scores with a constant probability p. How many push-ups does he do in expectation in his game?

b) Luigi wants to show that he is better and wants to score m times in sequence. After each miss, he performs 10 push-ups as well, and then tries again to score m times in a row. How many push-ups does Luigi in expectation, assuming he also scores with a constant probability p?

c) Trudy accepts Luigi's game and tries to score $m = 3$ times in a row. But Trudy is a bit lazy and gives up as soon as she has missed two times in a row. Trudy scores with constant probability $p = 0.5$.

(i) What is the probability that Trudy scores $m = 3$ times in a row? What is the probability that she gives up?

(ii) How many push-ups does Trudy do in expectation?

Hint:
$$\sum_{i=1}^{\infty} i \cdot q^{i-1} = \frac{1}{(1-q)^2} \qquad \text{for } |q| < 1$$