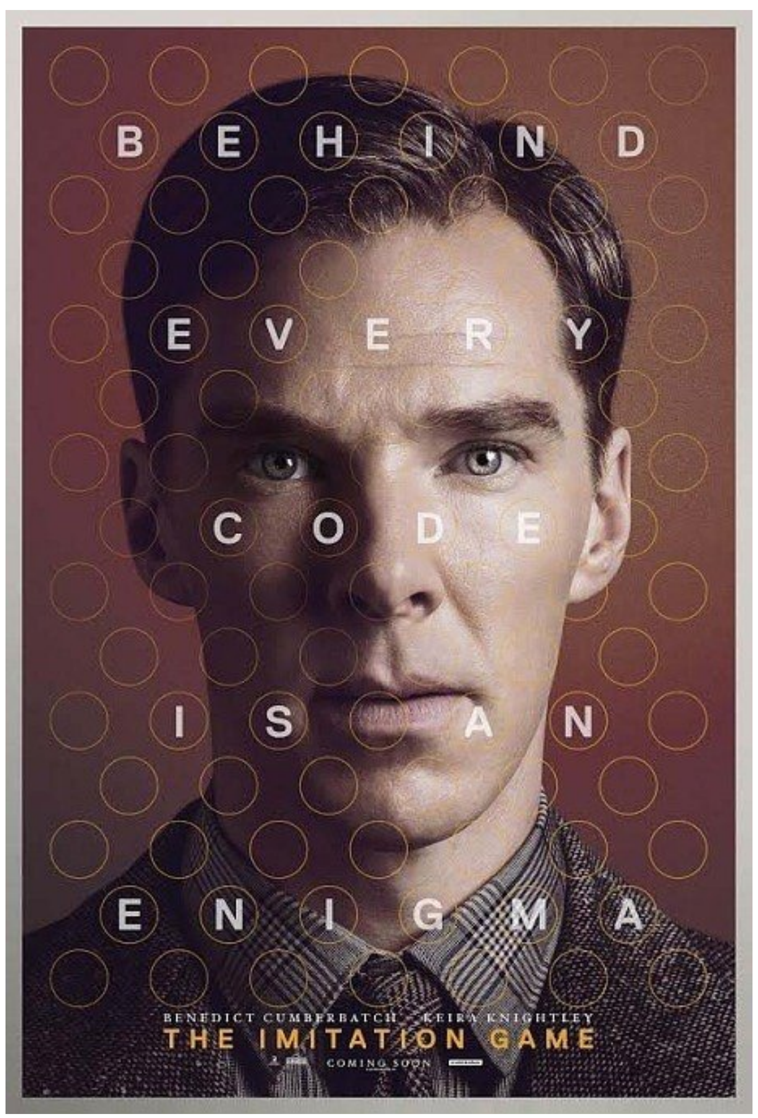
Automata & languages

A primer on the Theory of Computation



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Part 5 out of 5

Last week was all about

Context-Free Languages

Context-Free Languages

a superset of Regular Languages

Example $\{0^n 1^n \mid n \geq 0\}$ is a CFL but not a RL

We saw the concept of

Context-Free Grammars

$$(\mathcal{V}, \Sigma, \mathcal{R}, \mathcal{S})$$

- $\mathcal{V}$: variables
- Σ: alphabet
- $\mathcal{R}$: fine set of rules
- $\mathcal{S}$: start symbol

$$v \to w$$

$v \in \mathcal{V}$

$w \in (\Sigma_\epsilon \cup \mathcal{V})^*$

CFG's: Proving Correctness

- The recursive nature of CFG's means that they are especially amenable to correctness proofs.

- For example let's consider again our grammar

$$G = (S \rightarrow \varepsilon \mid ab \mid ba \mid aSb \mid bSa \mid SS)$$

- We claim that L(G) = L = $\{ x \in \{a,b\}^* \mid n_a(x) = n_b(x) \}$,
  where $n_a(x)$ is the number of *a*'s in *x*, and $n_b(x)$ is the number of *b*'s.

- *Proof*: To prove that L = L(G) is to show both inclusions:
  - *i.* $L \subseteq L(G)$: Every string in L can be generated by *G*.
  - *ii.* $L \supseteq L(G)$: *G* only generate strings of L.

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  i. $L \subseteq L(G)$: Every string in L can be generated by *G*.
  ii. $L \supseteq L(G)$: *G* only generate strings of L.

Part *ii.* is easy (see why?), so we'll concentrate on part *i.*

Proving $L \subseteq L(G)$

- $L \subseteq L(G)$: Show that every string x with the same number of a's as b's is generated by G. Prove by induction on the length $n = |x|$.

- Base case: The empty string is derived by $S \rightarrow \varepsilon$

- Inductive hypothesis:
  Assume that G generates all strings of equal number of a's and b's of (even) length up to n.

  Consider any string of length $n+2$. There are essentially 4 possibilities:
  1. *awb*
  2. *bwa*
  3. *awa*
  4. *bwb*

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  4. *bwb*

  Given $S \Rightarrow^* w$, *awb* and *bwa* are generated from *w* using the rules $S \rightarrow aSb$ and $S \rightarrow bSa$ (induction hypothesis)

Proving $L \subseteq L(G)$

- Inductive hypothesis:

  Now, consider a string like *awa*. For it to be in *L* requires that *w* isn't in *L* as *w* needs to have 2 more *b*'s than *a*'s.

  – Split *awa* as follows: $_0a_1$ ... $_{-1}a_0$
  where the subscripts after a prefix *v* of *awa* denotes $n_a(v) - n_b(v)$

  – Think of this as counting starting from *0*.
  Each *a* adds *1*. Each *b* subtracts *1*. At the end, we should be at *0.*

Somewhere along the string (in *w*), the counter crosses 0 (more b's)

Proving $L \subseteq L(G)$

- Inductive hypothesis:

  Somewhere along the string (in *w*), the counter crosses 0:

  u

  $_0a_1 \ldots \quad {}_{-1}x_0 \; y \ldots \quad {}_{-1}a_0$ with $x, y \in \{a, b\}$

  v

  – *u* and *v* have an equal number of *a's* and *b*'s and are shorter than *n*.
  – Given $S \Rightarrow^* u$ and $S \Rightarrow^* v$, the rule $S \rightarrow SS$ generates *awa* = *uv* (induction hypothesis)

  – The same argument applies for strings like *bwb*

As for Regular Languages,

Context-Free Languages are recognized by “machines”

Language	Regular	Context-Free
Machine	DFA/NFA	PDA

Push-Down Automatas are pretty similar to DFAs

alphabet

start state

$$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

states

accepting states

Push-Down Automatas are pretty similar to DFAs except for... the stack

stack alphabet

$$M = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

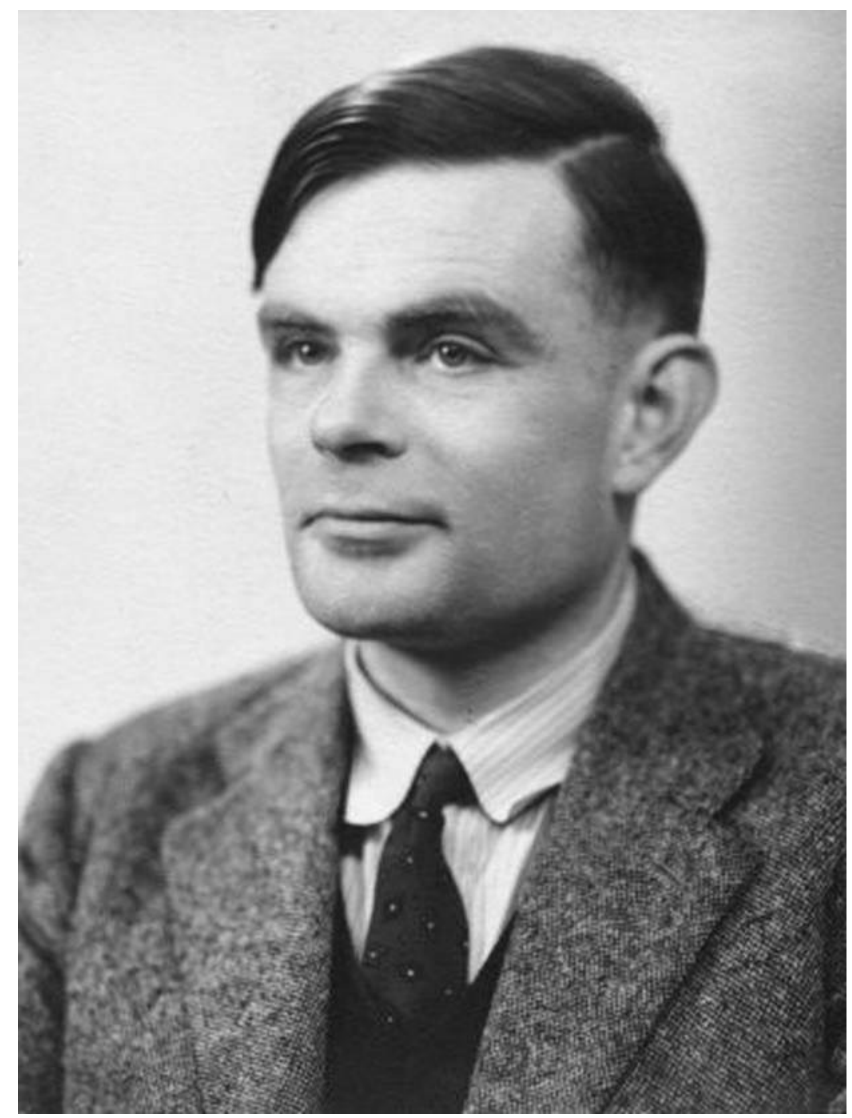
transition function

$$Q \times \Sigma_\epsilon \times \Gamma_\epsilon \rightarrow P(Q \times \Gamma_\epsilon)$$

This week, we'll see that

computers are not limitless

Alan Turing (1912-1954)



Some problems

cannot be solved

by a computer

(no matter its power)

But before that, we'll prove

some extra properties about Context-Free Languages

Today's plan

Thu Oct 18

1. PDA $\asymp$ CFG
2. Pumping lemma for CFL
3. Turing Machines

Even smarter automata…

- Even though the PDA is more powerful than the FA, it is still really stupid, since it doesn't understand a lot of important languages.

- Let's try to make it more powerful by adding a second stack
  - You can push or pop from either stack, also there's still an input string
  - Clearly there are quite a few "implementation details"
  - It seems at first that it doesn't help a lot to add a second stack, but…

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- Lemma: A PDA with two stacks is as powerful as a machine which operates on an infinite tape (restricted to read/write only "current" tape cell at the time – known as "Turing Machine").
  - Still that doesn't sound very exciting, does it…?!?

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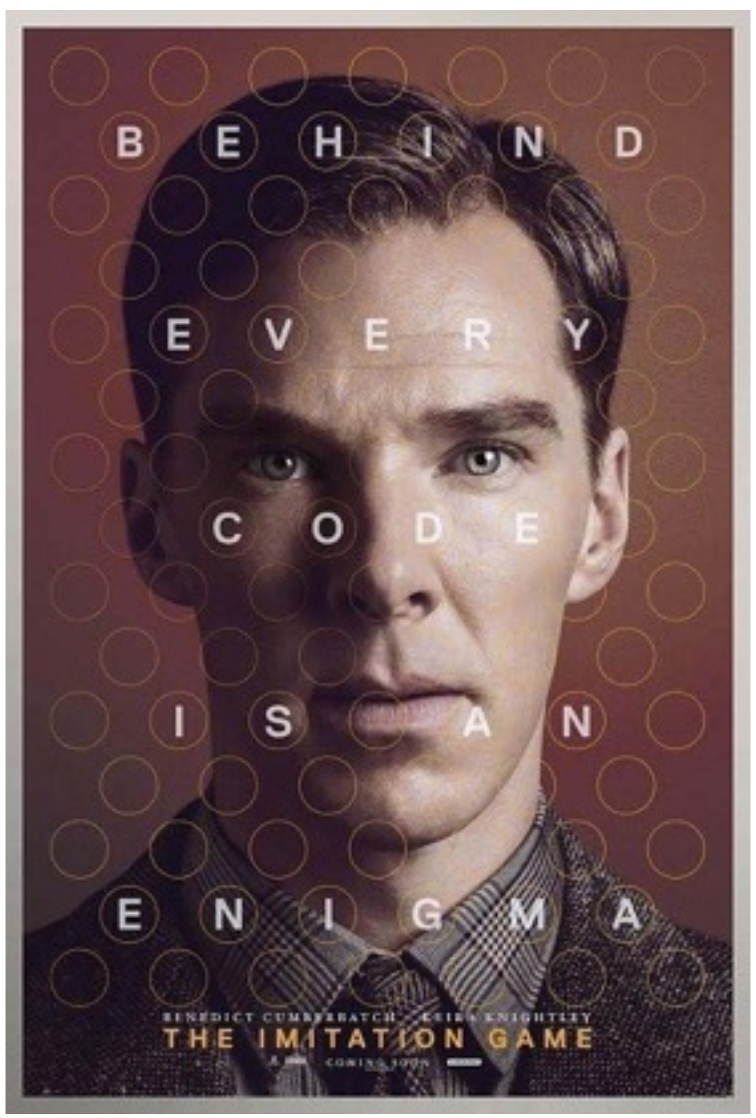

regular language

context-free language

Part 3 turing machine

Turing Machine

- A Turing Machine (TM) is a device with a finite amount of *read-only* "*hard*" memory (states), and an unbounded amount of read/write tape-memory. There is no separate input. Rather, the input is assumed to reside on the tape at the time when the TM starts running.

- Just as with Automata, TM's can either be input/output machines (compare with Finite State Transducers), or yes/no decision machines.

Turing Machine: Example Program

- Sample Rules:
  - If read 1, write 0, go right, repeat.
  - If read 0, write 1, HALT!
  - If read □, write 1, HALT! (the symbol □ stands for the blank cell)

- Let’s see how these rules are carried out on an input with the *reverse* binary representation of 47:

1	1	1	1	0	1				

Turing Machine: Formal Definition

- Definition: A Turing machine (TM) consists of a 7-tuple $M = (Q, \mathbf{\Sigma}, \mathbf{\Gamma}, \boldsymbol{\delta}, q_0, q_{acc}, q_{rej})$.
  - Q, Σ, and q_0, are the same as for an FA.
  - q_{acc} and q_{rej} are accept and reject states, respectively.
  - Γ is the tape alphabet which necessarily contains the blank symbol •, as well as the input alphabet Σ.
  - δ is as follows:

$$\delta : (Q - \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$$

  - Therefore given a non-halt state p, and a tape symbol x, $\delta(p,x) = (q,y,\text{D})$ means that TM goes into state q, replaces x by y, and the tape head moves in direction D (left or right).

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- A string x is accepted by M if after being put on the tape with the Turing machine head set to the left-most position, and letting M run, M eventually enters the accept state. In this case w is an element of $L(M)$ – the language accepted by *M.*

Comparison

Device	Separate Input?	Read/Write Data Structure	Deterministic by default?
FA	Yes	None	Yes
PDA	Yes	LIFO Stack	No
TM	No	1-way infinite tape. 1 cell access per step.	Yes (but will also allow crashes)

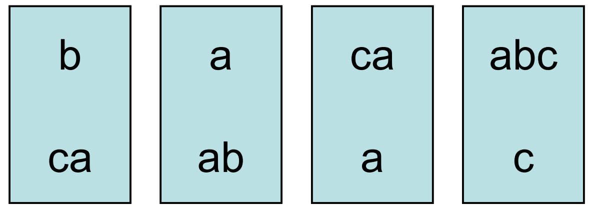
Turing Machine: Goals

- First Goal of Turing's Machine: A "computer" which is as powerful as any real computer/programming language
  - As powerful as C, or "Java++"
  - Can execute all the same algorithms / code
  - Not as fast though (move the head left and right instead of RAM)
  - Historically: A model that can compute anything that a human can compute. Before invention of electronic computers the term "computer" actually referred to a *person* who's line of work is to calculate numerical quantities!
  - This is known as the [Church-[Post-]] Turing thesis, 1936.

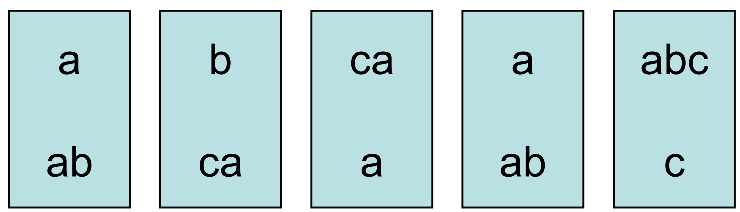
- Second Goal of Turing's Machine: And at the same time a model that is simple enough to actually prove interesting epistemological results.

Can a computer compute anything…?!?

- Given collection of dominos, e.g.



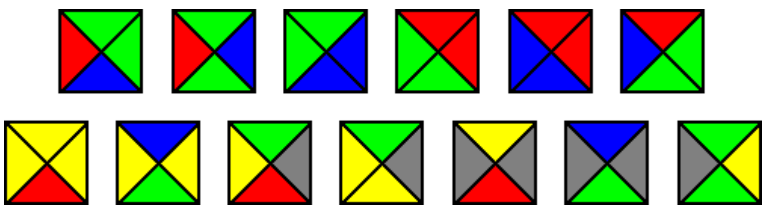
- Can you make a list of these dominos (repetitions are allowed) so that the top string equals the bottom string, e.g.



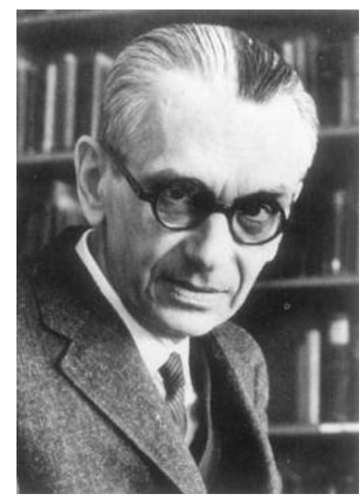
- This problem is known as Post-Correspondance-Problem.
- It is provably unsolvable by computers!

Also the Turing Machine (the Computer) is limited

- Similary it is undecidable whether you can cover a floor with a given set of floor tiles (famous examples are Penrose tiles or Wang tiles)



- Examples are leading back to Kurt Gödel's incompleteness theorem
  - "Any powerful enough axiomatic system will allow for propositions that are undecidable."



Decidability

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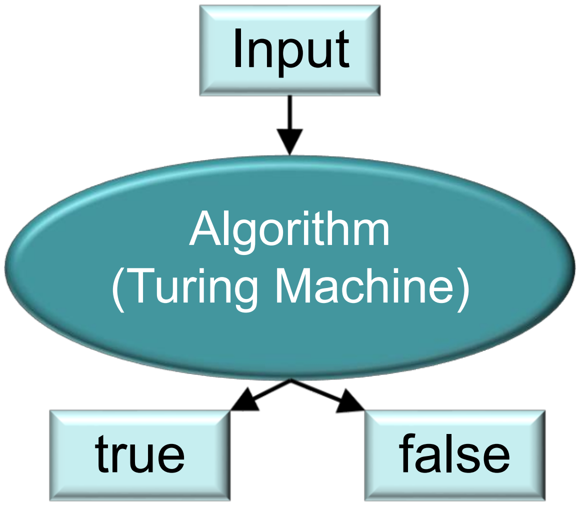

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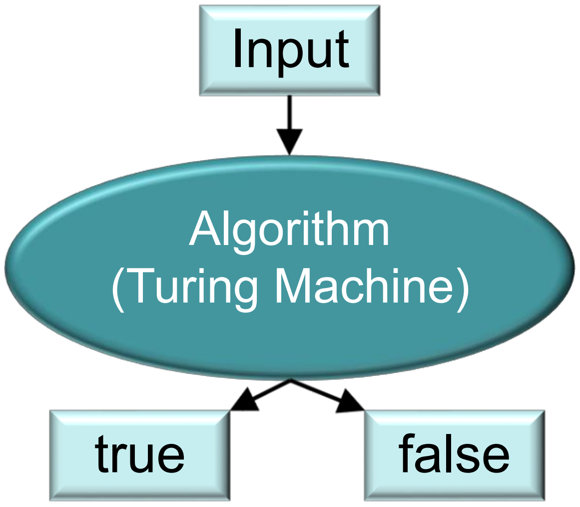
- A more general class are the semi-decidable problems, for which the algorithm must only terminate in finite time in either the true or the false branch, but not the other.



Halting Problem

- The halting problem is a famous example of an undecidable (semi-decidable) problem. Essentially, you cannot write a computer program that decides whether another computer program ever terminates (or has an infinite loop) on some given input.

- In pseudo code, we would like to have:

```
procedure halting(program, input) {
    if program(input) terminates
    then return true
    else return false
}
```

Halting Problem: Proof

- Now we write a little wrapper around our halting procedure

```
procedure test(program) {
    if halting(program,program)=true
    then loop forever
    else return
}
```

- Now we simply run: `test(test)`! Does it halt?!?

Excursion: P and NP

- P is the complexity class containing decision problems which can be solved by a Turing machine in time polynomial of the input size.

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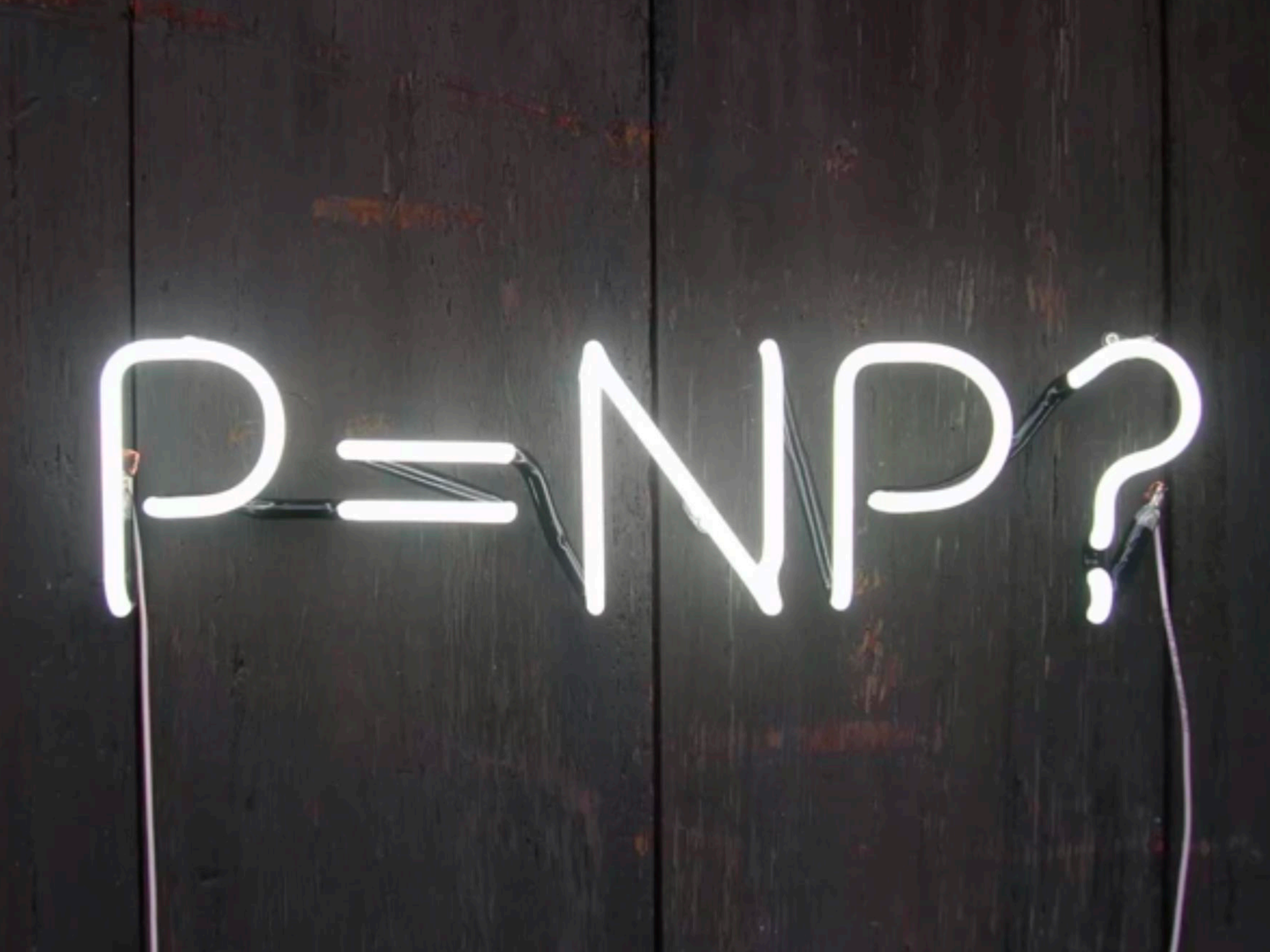
- NP is the class of decision problems solvable by a non-deterministic polynomial time Turing machine such that the machine answers "yes," if at least one computation path accepts, and answers "no," if all computation paths reject.
  - Informally, there is a Turing machine which can check the correctness of an answer in polynomial time.
  - E.g. one can check in polynomial time whether a traveling salesperson path connects *n* cities with less than a total distance *d*.

NP-complete problems

- An important notion in this context is the large set of NP-complete decision problems, which is a subset of NP and might be informally described as the "hardest" problems in NP.

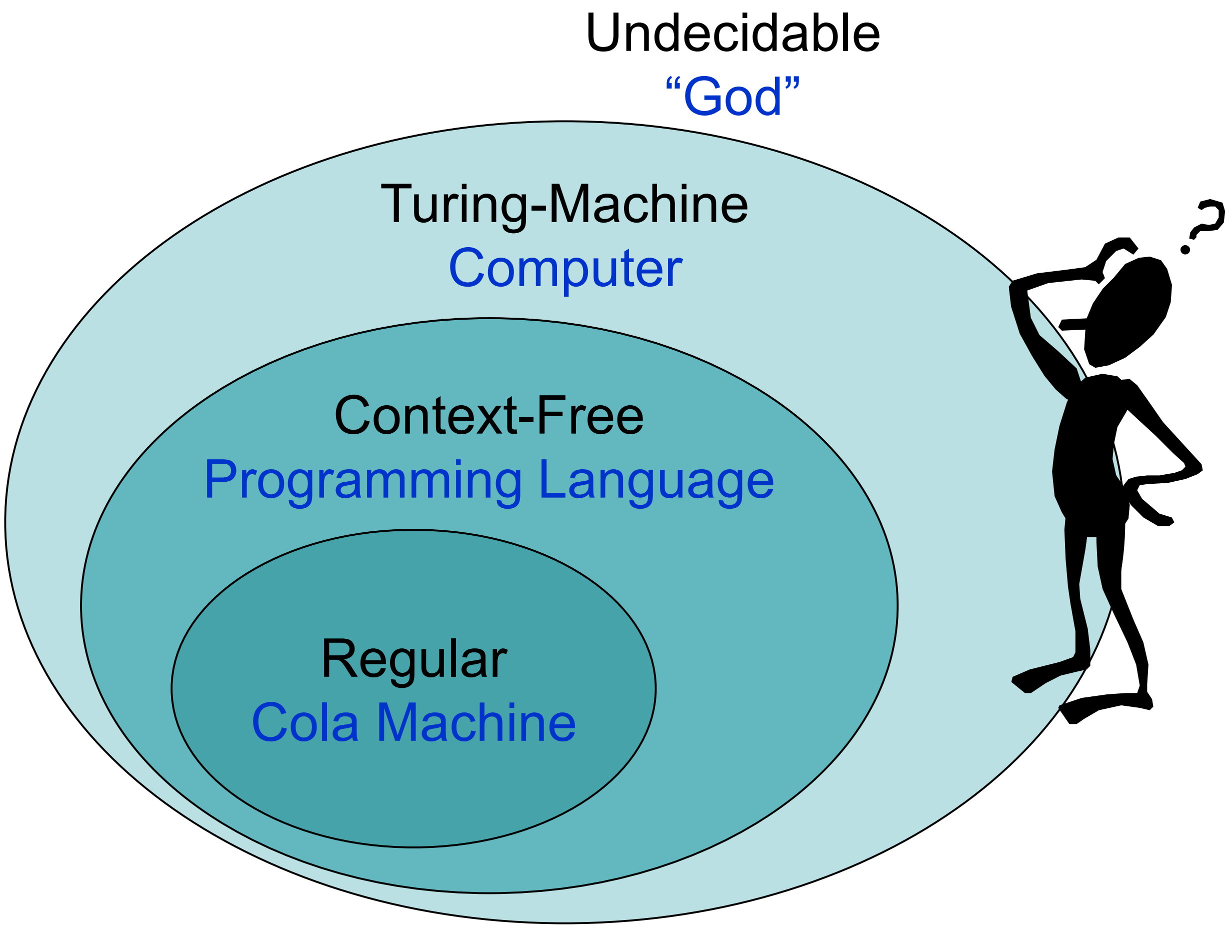
- If there is a polynomial-time algorithm for even one of them, then there is a polynomial-time algorithm for all the problems in NP.
  - E.g. Given a set of *n* integers, is there a non-empty subset which sums up to 0? This problem was shown to be NP-complete.
  - Also the traveling salesperson problem is NP-complete, or Tetris, or Minesweeper.


P=NP?

P vs. NP

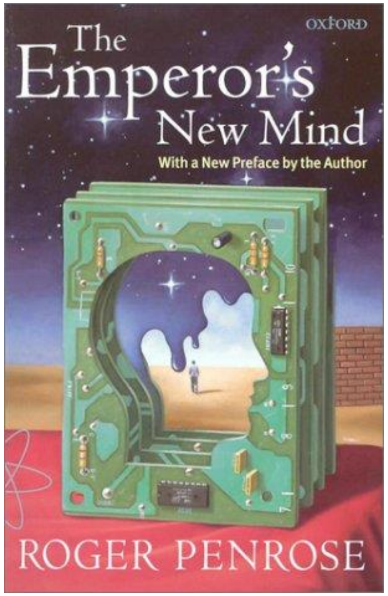
- One of the big questions in Math and CS: Is P = NP?
  - Or are there problems which cannot be solved in polynomial time.
  - Big practical impact (e.g. in Cryptography).
  - One of the seven $1M problems by the Clay Mathematics Institute of Cambridge, Massachusetts.

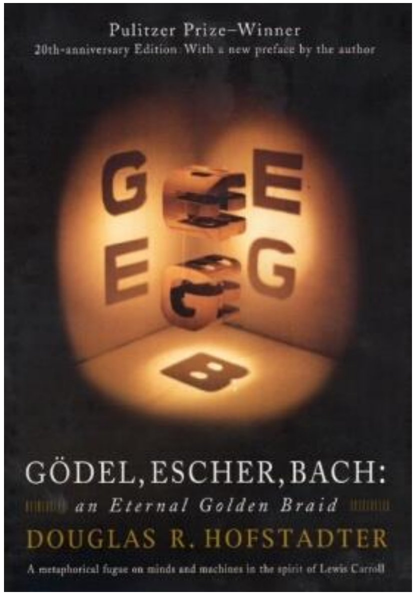
Summary (Chomsky Hierarchy)



Bedtime Reading

If you’re leaning towards “human = machine”





If you’re leaning towards “human ⊃ machine”