Chapter 11

Byzantine Agreement

In order to make flying safer, researchers studied possible failures of various sensors and machines used in airplanes. While trying to model the failures, they were confronted with the following problem: Failing machines did not just crash, instead they sometimes showed arbitrary behavior before stopping completely. With these insights researchers modeled failures as arbitrary failures, not restricted to any patterns.

Definition 11.1 (Byzantine)**.** *A node which can have arbitrary behavior is called* ***byzantine****. This includes "anything imaginable", e.g., not sending any messages at all, or sending different and wrong messages to different neighbors, or lying about the input value.*

Remarks:

- Byzantine behavior also includes collusion, i.e., all byzantine nodes are being controlled by the same adversary.
- We assume that any two nodes communicate directly, and that no node can forge an incorrect sender address. This is a requirement, such that a single byzantine node cannot simply impersonate all nodes!
- We call non-byzantine nodes *correct* nodes.

Definition 11.2 (Byzantine Agreement)**.** *Finding consensus as in Definition 8.1 in a system with byzantine nodes is called* ***byzantine agreement****. An algorithm is f-resilient if it still works correctly with f byzantine nodes.*

Remarks:

- As for consensus (Definition 8.1) we also need agreement, termination and validity. Agreement and termination are straight-forward, but what about validity?

11.1 Validity

Definition 11.3 (Any-Input Validity)**.** *The decision value must be the input value of* ***any*** *node.*

Remarks:

- This is the validity definition we used for consensus, in Definition 8.1.
- Does this definition still make sense in the presence of byzantine nodes? What if byzantine nodes lie about their inputs?
- We would wish for a validity definition which differentiates between byzantine and correct inputs.

Definition 11.4 (Correct-Input Validity)**.** *The decision value must be the input value of a* ***correct*** *node.*

Remarks:

- Unfortunately, implementing correct-input validity does not seem to be easy, as a byzantine node following the protocol but lying about its input value is indistinguishable from a correct node. Here is an alternative.

Definition 11.5 (All-Same Validity)**.** *If* ***all*** *correct nodes start with the same input v, the decision value must be v.*

Remarks:

- If the decision values are binary, then correct-input validity is induced by all-same validity.
- If the input values are not binary, but for example from sensors that deliever values in $\mathbb{R}$, all-same validity is in most scenarios not really useful.

Definition 11.6 (Median Validity)**.** *If the input values are orderable, e.g. $v \in \mathbb{R}$, byzantine outliers can be prevented by agreeing on a value close to the* ***median*** *of the correct input values – how close depends on the number of byzantine nodes f.*

Remarks:

- Is byzantine agreement possible? If yes, with what validity condition?
- Let us try to find an algorithm which tolerates 1 single byzantine node, first restricting to the so-called synchronous model.

Model 11.7 (synchronous)**.** *In the* ***synchronous model****, nodes operate in synchronous rounds. In each round, each node may send a message to the other nodes, receive the messages sent by the other nodes, and do some local computation.*

Definition 11.8 (synchronous runtime)**.** *For algorithms in the synchronous model, the* ***runtime*** *is simply the number of rounds from the start of the execution to its completion in the worst case (every legal input, every execution scenario).*

11.2 How Many Byzantine Nodes?

Algorithm 11.9 Byzantine Agreement with $f = 1$.

1: Code for node u, with input value x:

Round 1

2: Send $\texttt{tuple}(u, x)$ to all other nodes
3: Receive $\texttt{tuple}(v, y)$ from all other nodes v
4: Store all received $\texttt{tuple}(v, y)$ in a set S_u

Round 2

5: Send set S_u to all other nodes
6: Receive sets S_v from all nodes v
7: T = set of $\texttt{tuple}(v, y)$ seen in at least two sets S_v, including own S_u
8: Let $\texttt{tuple}(v, y) \in T$ be the tuple with the smallest value y
9: Decide on value y

Remarks:

- Byzantine nodes may not follow the protocol and send syntactically incorrect messages. Such messages can easily be deteced and discarded. It is worse if byzantine nodes send syntactically correct messages, but with a bogus content, e.g., they send different messages to different nodes.
- Some of these mistakes cannot easily be detected: For example, if a byzantine node sends different values to different nodes in the first round; such values will be put into S_u. However, some mistakes can and must be detected: Observe that all nodes only relay information in Round 2, and do not say anything about their own value. So, if a byzantine node sends a set S_v which contains a $\texttt{tuple}(v, y)$, this tuple must be removed by u from S_v upon receiving it (Line 6).
- Recall that we assumed that nodes cannot forge their source address; thus, if a node receives $\texttt{tuple}(v, y)$ in Round 1, it is guaranteed that this message was sent by v.

Lemma 11.10. *If $n \geq 4$, all correct nodes have the same set T.*

Proof. With $f = 1$ and $n \geq 4$ we have at least 3 correct nodes. A correct node will see every correct value at least twice, once directly from another correct node, and once through the third correct node. So all correct values are in T. If the byzantine node sends the same value to at least 2 other (correct) nodes, all correct nodes will see the value twice, so all add it to set T. If the byzantine node sends all different values to the correct nodes, none of these values will end up in any set T. □

Theorem 11.11. *Algorithm 11.9 reaches byzantine agreement if $n \geq 4$.*

Proof. We need to show agreement, any-input validity and termination. With Lemma 11.10 we know that all correct nodes have the same set T, and therefore

agree on the same minimum value. The nodes agree on a value proposed by any node, so any-input validity holds. Moreover, the algorithm terminates after two rounds. □

Remarks:

- If $n > 4$ the byzantine node can put multiple values into T.
- Algorithm 11.9 only provides any-input agreement, which is questionable in the byzantine context. One can achieve all-same validity by choosing the smallest value that occurs at least twice, if a value appears at least twice.
- The idea of this algorithm can be generalized for any f and $n > 3f$. In the generalization, every node sends in every of $f + 1$ rounds all information it learned so far to all other nodes. In other words, message size increases exponentially with f.
- Does Algorithm 11.9 also work with $n = 3$?

Theorem 11.12. *Three nodes cannot reach byzantine agreement with all-same validity if one node among them is byzantine.*

Proof. We will assume that the three nodes satisfy all-same validity and show that they will violate the agreement condition under this assumption.

In order to achieve all-same validity, nodes have to deterministically decide for a value x if it is the input value of every correct node. Recall that a Byzantine node which follows the protocol is indistinguishable from a correct node. Assume a correct node sees that $n - f$ nodes including itself have an input value x. Then, by all-same validity, this correct node must deterministically decide for x.

In the case of three nodes ($n - f = 2$) a node has to decide on its own input value if another node has the same input value. Let us call the three nodes u, v and w. If correct node u has input 0 and correct node v has input 1, the byzantine node w can fool them by telling u that its value is 0 and simultaneously telling v that its value is 1. By all-same validity, this leads to u and v deciding on two different values, which violates the agreement condition. Even if u talks to v, and they figure out that they have different assumptions about w's value, u cannot distinguish whether w or v is byzantine. □

Theorem 11.13. *A network with n nodes cannot reach byzantine agreement with $f \geq n/3$ byzantine nodes.*

Proof. Assume (for the sake of contradiction) that there exists an algorithm A that reaches byzantine agreement for n nodes with $f \geq \lceil n/3 \rceil$ byzantine nodes. We will show that A cannot satisfy all-same validity and agreement simultaneously.

Let us divide the n nodes into three groups of size $n/3$ (either $\lfloor n/3 \rfloor$ or $\lceil n/3 \rceil$, if n is not divisible by 3). Assume that one group of size $\lceil n/3 \rceil \geq n/3$ contains only Byzantine and the other two groups only correct nodes. Let one group of correct nodes start with input value 0 and the other with input value 1. As in Lemma 11.12, the group of Byzantine nodes supports the input value of each of the node, so each correct node observes at least $n - f$ nodes who support its own input value. Because of all-same validity, every correct node has to deterministically decide on its own input value. Since the two groups of correct nodes had different input values, the nodes will decide on different values respectively, thus violating the agreement property. □

11.3 The King Algorithm

Algorithm 11.14 King Algorithm (for $f < n/3$)

x = my input value
for phase = 1 to $f + 1$ **do**
  Round 1
  Broadcast `value`(x)
  Round 2
  if some `value`(y) received at least $n - f$ times **then**
    Broadcast `propose`(y)
  end if
  if some `propose`(z) received more than f times **then**
    $x = z$
  end if
  Round 3
  Let node v_i be the predefined king of this phase i
  The king v_i broadcasts its current value w
  if received strictly less than $n - f$ `propose`(y) **then**
    $x = w$
  end if
end for

Lemma 11.15. *Algorithm 11.14 fulfills the all-same validity.*

Proof. If all correct nodes start with the same value, all correct nodes propose it in Round 2. All correct nodes will receive at least $n - f$ proposals, i.e., all correct nodes will stick with this value, and never change it to the king's value. This holds for all phases. □

Lemma 11.16. *If a correct node proposes x, no other correct node proposes y, with $y \neq x$, if $n > 3f$.*

Proof. Assume (for the sake of contradiction) that a correct node proposes value x and another correct node proposes value y. Since a good node only proposes a value if it heard at least $n-f$ `value` messages, we know that both nodes must have received their value from at least $n-2f$ distinct correct nodes (as at most f nodes can behave byzantine and send x to one node and y to the other one). Hence, there must be a total of at least $2(n-2f) + f = 2n - 3f$ nodes in the system. Using $3f < n$, we have $2n - 3f > n$ nodes, a contradiction. □

Lemma 11.17. *There is at least one phase with a correct king.*

Proof. There are $f + 1$ phases, each with a different king. As there are only f byzantine nodes, one king must be correct. □

Lemma 11.18. *After a round with a correct king, the correct nodes will not change their values v anymore, if $n > 3f$.*

Proof. If all correct nodes change their values to the king's value, all correct nodes have the same value. If some correct node does not change its value to the king's value, it received a proposal at least $n - f$ times, therefore at least $n-2f$ correct nodes broadcasted this proposal. Thus, all correct nodes received it at least $n - 2f > f$ times (using $n > 3f$), therefore all correct nodes set their value to the proposed value, including the correct king. Note that only one value can be proposed more than f times, which follows from Lemma 11.16. With Lemma 11.15, no node will change its value after this round. □

Theorem 11.19. *Algorithm 11.14 solves byzantine agreement.*

Proof. The king algorithm reaches agreement as either all correct nodes start with the same value, or they agree on the same value latest after the phase where a correct node was king according to Lemmas 11.17 and 11.18. Because of Lemma 11.15 we know that they will stick with this value. Termination is guaranteed after $3(f + 1)$ rounds, and all-same validity is proved in Lemma 11.15. □

Remarks:

- Algorithm 11.14 requires $f + 1$ predefined kings. We assume that the kings (and their order) are given. Finding the kings indeed would be a byzantine agreement task by itself, so this must be done before the execution of the King algorithm.
- Do algorithms exist which do not need predefined kings? Yes, see Section 11.5.
- Can we solve byzantine agreement (or at least consensus) in less than $f + 1$ rounds?

11.4 Lower Bound on Number of Rounds

Theorem 11.20. *A synchronous algorithm solving consensus in the presence of f crashing nodes needs at least $f+1$ rounds, if nodes decide for the minimum seen value.*

Proof. Let us assume (for the sake of contradiction) that some algorithm A solves consensus in f rounds. Some node u_1 has the smallest input value x, but in the first round u_1 can send its information (including information about its value x) to only some other node u_2 before u_1 crashes. Unfortunately, in the second round, the only witness u_2 of x also sends x to exactly one other node u_3 before u_2 crashes. This will be repeated, so in round f only node u_{f+1} knows about the smallest value x. As the algorithm terminates in round f, node u_{f+1} will decide on value x, all other surviving (correct) nodes will decide on values larger than x. □

Remarks:

- A general proof without the restriction to decide for the minimum value exists as well.
- Since byzantine nodes can also just crash, this lower bound also holds for byzantine agreement, so Algorithm 11.14 has an asymptotically optimal runtime.
- So far all our byzantine agreement algorithms assume the synchronous model. Can byzantine agreement be solved in the asynchronous model?

11.5 Asynchronous Byzantine Agreement

Algorithm 11.21 Asynchronous Byzantine Agreement (Ben-Or, for $f < n/10$)

1: $x_u \in \{0,1\}$ ◁ input bit
2: r = 1 ◁ round
3: decided = false
4: Broadcast `propose`(x_u,r)
5: **repeat**
6:   Wait until $n - f$ `propose` messages of current round r arrived
7:   **if** at least $n/2 + 3f + 1$ `propose` messages contain same value x **then**
8:     $x_u = x$, decided = true
9:   **else if** at least $n/2 + f + 1$ `propose` messages contain same value x **then**
10:     $x_u = x$
11:   **else**
12:     choose x_u randomly, with $Pr[x_u = 0] = Pr[x_u = 1] = 1/2$
13:   **end if**
14:   r = r + 1
15:   Broadcast `propose`(x_u,r)
16: **until** decided (see Line 8)
17: decision = x_u

Lemma 11.22. *Let a correct node choose value x in Line 10, then no other correct node chooses value $y \neq x$ in Line 10.*

Proof. For the sake of contradiction, assume that both 0 and 1 are chosen in Line 10. This means that both 0 and 1 had been proposed by at least $n/2 + 1$ out of $n - f$ correct nodes. In other words, we have a total of at least $2 \cdot n/2 + 2 = n + 2 > n - f$ correct nodes. Contradiction! □

Theorem 11.23. *Algorithm 11.21 solves binary byzantine agreement as in Definition 11.2 for up to $f < n/10$ byzantine nodes.*

Proof. First note that it is not a problem to wait for $n - f$ propose messages in Line 6, since at most f nodes are byzantine. If all correct nodes have the same input value x, then all (except the f byzantine nodes) will propose the same value x. Thus, every node receives at least $n - 2f$ propose messages containing x. Observe that for $f < n/10$, we get $n - 2f > n/2 + 3f$ and the nodes will decide on x in the first round already. We have established all-same validity! If the correct nodes have different (binary) input values, the validity condition becomes trivial as any result is fine.

What about agreement? Let u be the first node to decide on value x (in Line 8). Due to asynchrony another node v received messages from a different subset of the nodes, however, at most f senders may be different. Taking into account that byzantine nodes may lie (send different propose messages to different nodes), f additional propose messages received by v may differ from those received by u. Since node u had at least $n/2 + 3f + 1$ propose messages with value x, node v has at least $n/2 + f + 1$ propose messages with value x. Hence every correct node will propose x in the next round, and then decide on x.

So we only need to worry about termination: We have already seen that as soon as one correct node terminates (Line 8) everybody terminates in the next round. So what are the chances that some node u terminates in Line 8? Well, we can hope that all correct nodes randomly propose the same value (in Line 12). Maybe there are some nodes not choosing randomly (entering Line 10 instead of 12), but according to Lemma 11.22 they will all propose the same.

Thus, at worst all $n - f$ correct nodes need to randomly choose the same bit, which happens with probability $2^{-(n-f)+1}$. If so, all correct nodes will send the same propose message, and the algorithm terminates. So the expected running time is exponential in the number of nodes n in the worst case. □

Remarks:

- This Algorithm is a proof of concept that asynchronous byzantine agreement can be achieved. Unfortunately this algorithm is not useful in practice, because of its runtime.
- Note that for $f \in O(\sqrt{n})$, the probability for some node to terminate in Line 8 is greater than some positive constant. Thus, the Ben-Or algorithm terminates within expected constant number of rounds for small values of f.

Chapter Notes

The project which started the study of byzantine failures was called SIFT and was founded by NASA [WLG+78], and the research regarding byzantine agreement started to get significant attention with the results by Pease, Shostak, and Lamport [PSL80, LSP82]. In [PSL80] they presented the generalized version of Algorithm 11.9 and also showed that byzantine agreement is unsolvable for $n \leq 3f$. The algorithm presented in that paper is nowadays called *Exponential Information Gathering (EIG)*, due to the exponential size of the messages.

There are many algorithms for the byzantine agreement problem. For example the Queen Algorithm [BG89] which has a better runtime than the King algorithm [BGP89], but tolerates less failures. That byzantine agreement requires at least $f + 1$ many rounds was shown by Dolev and Strong [DS83], based on a more complicated proof from Fischer and Lynch [FL82].

While many algorithms for the synchronous model have been around for a long time, the asynchronous model is a lot harder. The only results were by

Ben-Or and Bracha. Ben-Or [Ben83] was able to tolerate $f < n/5$. Bracha [BT85] improved this tolerance to $f < n/3$.

Nearly all developed algorithms only satisfy all-same validity. There are a few exceptions, e.g., correct-input validity [FG03], available if the initial values are from a finite domain, median validity [SW15, MW18, DGM+11] if the input values are orderable, or values inside the convex hull of all correct input values [VG13, MH13, MHVG15] if the input is multidimensional.

Before the term *byzantine* was coined, the terms Albanian Generals or Chinese Generals were used in order to describe malicious behavior. When the involved researchers met people from these countries they moved – for obvious reasons – to the historic term byzantine [LSP82].

Hat tip to Peter Robinson for noting how to improve Algorithm 11.9 to all-same validity. This chapter was written in collaboration with Barbara Keller.

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