Eidgenössische Technische Hochschule Zürich
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Networked Systems Group (NSG)

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based on Prof. L. Thiele's material

Discrete Event Systems

Solution to Exercise Sheet 12

1 Structural Properties of Petri Nets and Token Game

a) The Pre and Post sets of a transition are defined as follows:

- Pre set: $\bullet t := \{p \mid (p,t) \in F\}$
- Post set: $t\bullet := \{p \mid (t,p) \in F\}$,

where F is the flow set, i.e., the set of place/transition and transition/place arcs. The Pre and Post sets of a place are defined analogously.

For the Petri net N_1 we obtain the following sets:

$$\begin{aligned} \bullet t_5 &= \{p_5, p_9\}, & t_5\bullet &= \{p_6\} \\ \bullet t_8 &= \{p_8\}, & t_8\bullet &= \{p_{10}, p_5\} \\ \bullet p_3 &= \{t_2\}, & p_3\bullet &= \{t_3\} \end{aligned}$$

b) A transition is enabled if all places in its Pre set contain enough tokens. In the case of N_1, which has only unweighted edges, one token per place suffices. When t_2 fires, it consumes one token out of each place in the Pre set of t_2 and produces one token on each place in the Post set of t_2. Hence, the firing of t_2 produces one token on place p_3 and p_9 each, while the token in p_2 is consumed.

As a result, $\mathbf{t_5}$ is enabled because both p_9 and p_5 hold one token. However, t_3 is not enabled because p_3 contains a token but p_{10} does not.

c) Before t_2 fires there are two tokens in N_1, one in p_2 and one in p_5. After the firing, there is one token in places p_3, p_9 and p_5, hence 3 tokens in total.

d) A token traverses the upper cycle until t_2 fires. Then one token remains on p_3 and waits, and another one is produced in p_9, which enables transition t_5. When t_5 consumes the tokens on p_9 and p_5 and produces a token on p_6, this one traverses the lower cycle until t_8 is enabled and fired. One token now remains on p_5 and waits, another is in p_{10} and enables t_3, because there is another token on p_3. Then one token traverses the upper cycle again until t_2 is enabled, and so on. This Petri net models two alternating processes.

This Petri net is clearly bounded, thus we can construct its reachability tree. Usually the states of Petri nets are denoted by vectors such that the i-th position in the vector indicates the number of tokens on place p_i of the Petri net, i.e., the marking of the graph. So, for example, the starting state $\vec{s}_0$ of N_1, in which the places p_1 and p_5 hold one token each, is denoted by $\vec{s}_0 = (1, 0, 0, 0, 1, 0, 0, 0, 0, 0)$.

For better legibility we denote the states in such a way that the index contains the places that hold a token in this state, for example $\vec{s}_0 = (1,0,0,0,1,0,0,0,0,0) \triangleq s_{1,5}$.

Then the reachability graph can also be written as,

$$\begin{tikzcd}
{} \arrow[r] & s_{1,5} \arrow[r, "t_1"] & s_{2,5} \arrow[r, "t_2"] & s_{3,5,9} \arrow[r, "t_5"] & s_{3,6} \arrow[r, "t_6"] & s_{3,7} \arrow[r, "t_7"] & s_{3,8} \arrow[r, "t_8"] & s_{3,5,10} \arrow[r, "t_3"] & s_{4,5} \arrow[lllllll, bend left, "t_4"]
\end{tikzcd}$$

2 Basic Properties of Petri Nets

A Petri net is k-bounded, if there is no fire sequence that makes the number of tokens in one place grow larger than k. It is obvious that Petri net N_2 is 1-bounded if $k \leq 1$. This holds because in the initial state there is only one token in the net, and in the case $k \leq 1$ no transition increases the number of tokens in N_2. If $k \geq 2$, the number of tokens in p_1 can grow infinitely large by repeatedly firing t_1, t_3 and t_4. So, the Petri net N_2 is unbounded for $k \geq 2$.

A Petri net is deadlock free if no fire sequence leads to a state in which no transition is enabled. If $k = 0$, N_2 is not deadlock-free. The fire sequence t_1, t_3, t_4 causes the only existing token to be consumed and hence, there is no enabled transition any more. For $k \geq 1$, however, no deadlock can occur.

3 Identifying a deadlock

a) There are an infinite number of blocking sequence: any number of cycles $t_{A0}t_{A1}t_{A2}$ and/or $t_{B0}t_{B1}t_{B2}$ terminated by either $t_{A0}t_{B0}$ or $t_{B0}t_{A0}$. It can be read directly from the marking graph below:

$$\begin{tikzcd}
 & & & (0,0,1,0,0,1,0,0) \arrow[ddll, bend right, "t_{A2}"'] \\
 & & (0,1,0,0,1,1,0,0) \arrow[ur, "t_{A1}"] \arrow[dr, "t_{B0}"] & \\
M_0 = & (1,0,0,1,1,1,0,0) \arrow[ur, "t_{A0}"] \arrow[dr, "t_{B0}"] & & (0,1,0,0,0,0,1,0) \\
 & & (1,0,0,1,0,0,1,0) \arrow[ur, "t_{A0}"] \arrow[dr, "t_{B1}"] & \\
 & & & (1,0,0,0,0,0,0,1) \arrow[uull, bend left, "t_{B2}"]
\end{tikzcd}$$

b) From the Petri net structure, we get:

$$W^+ = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, W^- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

$$A = W^+ - W^- = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

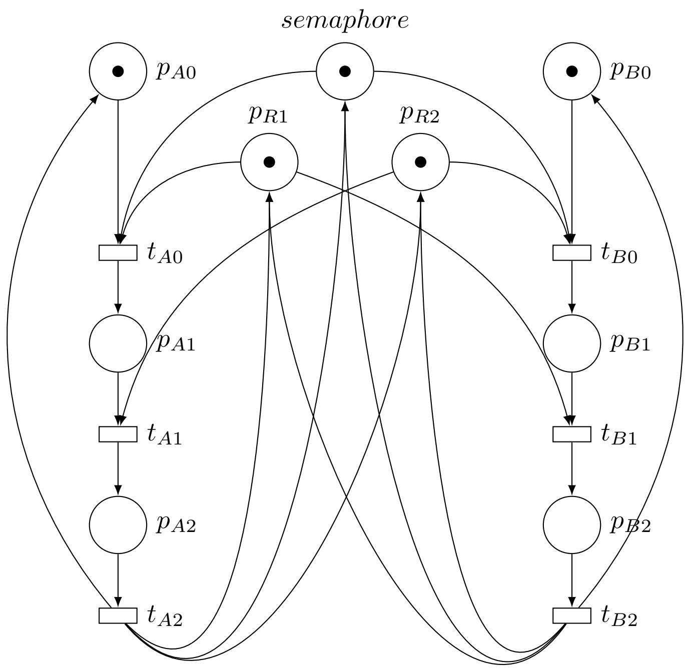
Consider the firing sequence $t_{A0}t_{B0}$. It entails:

$$M_{deadlock} = M_0 + A \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

As expected, we find again the blocking marking from the reachability graph of question **a)**.

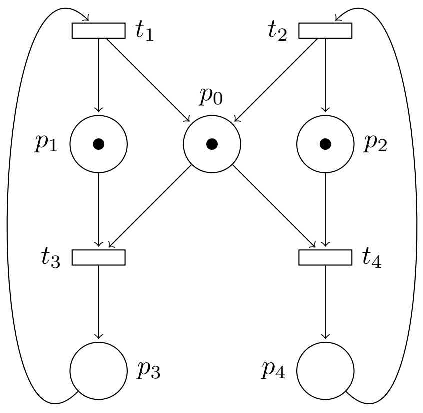
c) A deadlock state is a state at which no transition in enabled. Hence, one can use the upstream transition matrix W^- to assess whether or not a marking is blocking. It is the case if and only if the marking vector does not **cover** (i.e., is bigger or equal to) any column of W^-. Otherwise, it implies the transition associated to such column is enabled, and therefore this marking is not blocking.

d) In order to avoid such deadlock, it suffices to forbid both process to run concurrently. This can be solved easily using a semaphore, as illustrated thereafter:



4 From mutual exclusion to starvation

a) For each process we introduce two places (p_1, p_2, p_3 and p_4) representing the process within the normal program execution (p_1,p_2) as well as in the critical section (p_3,p_4). For each process, we have a token indicating which section of the program is currently executed. Additionally, we introduce a place p_0 representing the mutex variable. If the mutex variable is 0, then we have a token at p_0. We have to make sure that a process can only enter its critical section if there is a token at the mutex place. The resulting Petri net looks as follows.

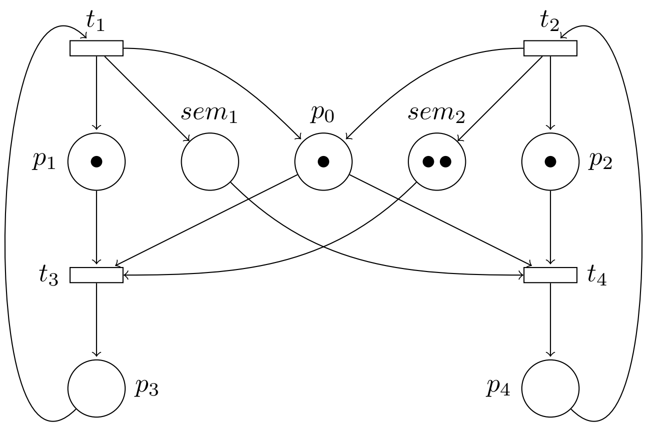


Assume that initially, both processes are in an non-critical section (in the Petri net, this is denoted by a token in place p_1 and p_2 respectively). A process can only enter its critical section (p_3/p_4) if there is a token at p_0. In this case, the token is consumed when entering the critical section. A new mutex token at p_0 is not created until the process leaves its critical section. Hence, both processes exclude each other mutually from the concurrent access to the critical section.

This is a classical benefit of Petri nets over other DES models. It models very efficiently the sharing of resources, the concurrency of processes, and so on...

b) In order to avoid starvation of either of the process, one option is to count the number of execution the each of them, or more precisely the difference between them. Assume that at initial state, none has been previously run. According to the specification, we can allow one to the process (say A) to run twice by creating a "counter-resource" with 2 tokens in the initial marking. Running the process A consumes one of these tokens. A new token is produced in this place on completion of process B. Doing that symmetrically (for each process) binds the number of executions of each process together...

Not so clear? Okay, just have a look to the net :-)



c) With such mechanism, A will not starve B. However, if for some reason, B does not execute anymore, then A will have to stop as well once it has two executions more than B. This would a pretty bad design in most cases.

The naive idea would be to say that "if both processes want to access the resource, they get it in turns".

5 Coverability tree and graph

Following the procedure from the lecture note, we can construct the following coverability tree:

$$\begin{tikzcd}
 & & & (1,\omega,0) & \\
M_0 = & (1,0,0) \arrow[r, "t_1"] & (1,\omega,0) \arrow[ur, "t_1"] \arrow[r, "t_2"] \arrow[dr, "t_3"] & (1,\omega,0) & (1,\omega,\omega) \\
 & & & (1,\omega,\omega) \arrow[ur, "t_1"] \arrow[r, "t_2"] \arrow[dr, "t_3"] & (1,\omega,\omega) \\
 & & & & (1,\omega,\omega)
\end{tikzcd}$$

One can merge the equivalent node and obtain the coverability graph:

$$\begin{tikzcd}
M_0 = & (1,0,0) \arrow[r, "t_1"] & (1,\omega,0) \arrow[r, "t_3"] \arrow[loop below, "{t_1, t_2}"] & (1,\omega,\omega) \arrow[loop below, "{t_1, t_2, t_3}"]
\end{tikzcd}$$

It follows that for this net with initial marking $(1, 0, 0)$, places p_2 and p_3 are unbounded.

6 Reachability Analysis for Petri Nets

a) Petri nets may possess infinite reachability graphs, e.g. N_2 with $k \geq 2$. If a marking is actually reachable in such a Petri net, the reachability check will eventually terminate. But if it is not reachable, the algorithm may not be able to determine reachability with absolute certainty (cf. halting problem).

Constructing a coverability tree or graph is guaranteed to terminate. It can be used to prove that a given marking is not reachable, in the case where the marking you are interested is **not covered by any** marking in the coverability tree/graph. However, this is not a sufficient to prove reachability in the general case: a marking may be covered by the coverability graph, and yet not being reachable.

b) We determine the incidence matrix of the Petri net as explained in the lecture.

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

We are interested in whether the state $\vec{s} = (101, 99, 4)$ is reachable from the initial state $\vec{s_0} = (1, 0, 0)$. If the equation system $\mathbf{A} \cdot \vec{f} = \vec{s} - \vec{s_0}$ has no solution, we know for sure that the state $\vec{s}$ is not reachable from s_0. "Unfortunately",

$$\begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 100 \\ 99 \\ 4 \end{pmatrix}$$

is satisfiable. Using linear algebra, the solutions to this system can be computed (here, $f_1 = Q, f_2 = Q - 306, f_3 = 207, f_4 = 203$, for any $Q \in \mathbb{N}$). If $\vec{s}$ is reachable from $\vec{s_0}$, the firing sequence will be of this form. However, there is no guarantee that it is actually feasible for the net! Ultimately, one has to look at the net and propose a suitable firing sequence (although the solution to the previous system of equations gives us the "shape" of the firing sequence we are looking for).

So, to prove that $\vec{s}$ is reachable from $\vec{s_0}$, we have to give a firing sequence through which we get from $\vec{s_0}$ to $\vec{s}$. Considering the Petri net, we can see that – starting from $\vec{s_0}$ – the number of tokens in p_1 increases by one after firing the sequence t_1, t_3, t_4. Repeating this for 203 times yields the state $(204, 0, 0)$. Firing t_1 for 103 more times, followed by firing t_3 for four times finally yields state $\vec{s}$.