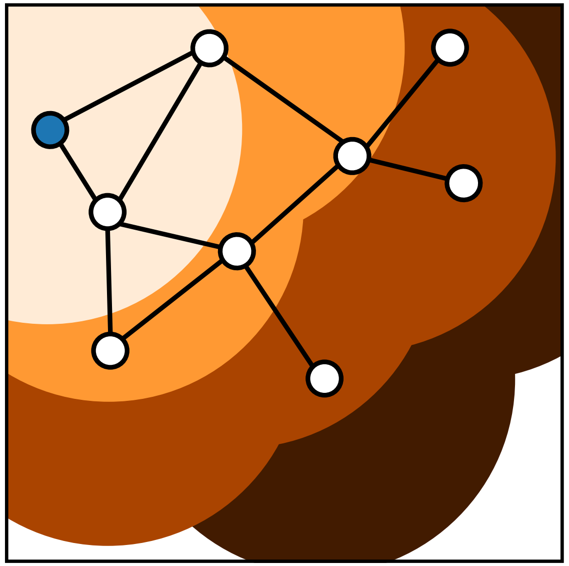
Discrete Event Systems
Verification of Finite Automata (Part 1)



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Most materials from Lothar Thiele and Romain Jacob

What are finite automata useful for?

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specification

- Digital circuits
- Protocols (e.g. BGP)

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synthesis of software or hardware

- Hardware components
- Network configurations

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verification

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Questions:

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- Do implementation and specification describe the same behavior?
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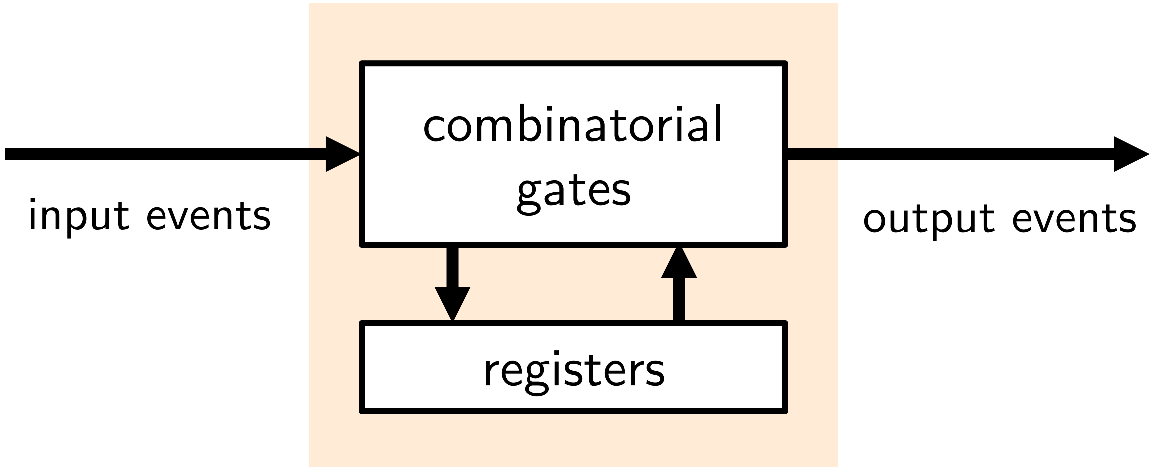
- Formal analysis (sometimes also called verification)
  - Formal (unambiguous) proof of correctness.

Verification of Finite Automata

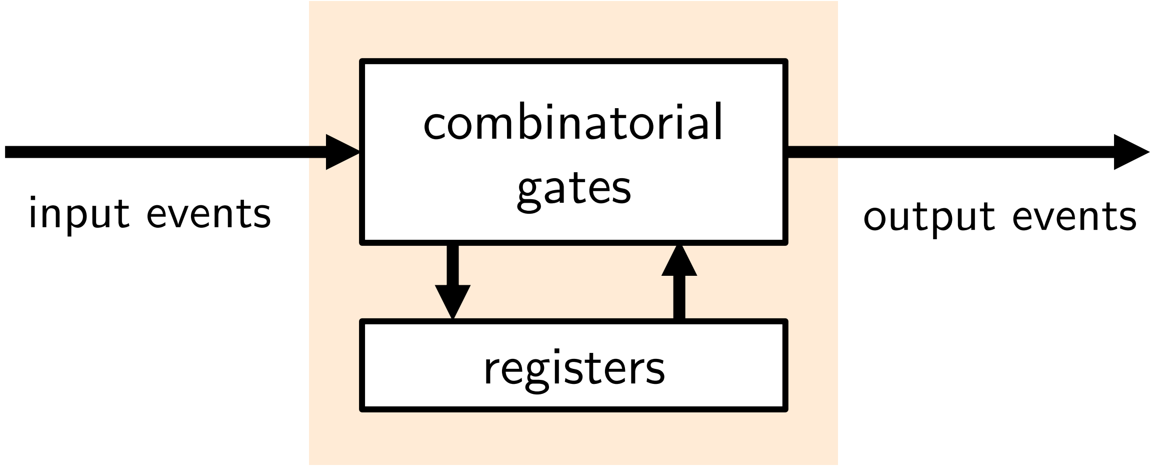
- Due to the finite number of states, proving properties of a finite state machine can be done by enumeration.
- As computer systems have finite memory, properties of processors (and embedded systems in general) could be shown in principle.



finite automaton

Verification of Finite Automata

- Due to the finite number of states, proving properties of a finite state machine can be done by enumeration.
- As computer systems have finite memory, properties of processors (and embedded systems in general) could be shown in principle.
- But is enumeration a reasonable approach in practice?



finite automaton

memory	number of states
8 Bit	256
32 Bit	4.10^9
1KBit	10^{300}
1MBit	$10^{300\,000}$
1GBit	$10^{300\,000\,000}$

atoms in the universe is about 10^{82}

Verification of Finite Automata

- There have been **major breakthroughs** in recent years on the verification of finite automata with very large state spaces. Prominent methods are based on
  - transformation to a Boolean Satisfiability (SAT) problem (not covered in this course) and
  - symbolic model checking via binary decision diagrams (covered in this course).

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  - symbolic model checking via binary decision diagrams (covered in this course).

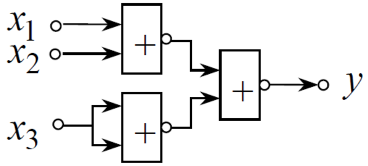
- **Symbolic model checking** is a method of verifying temporal properties of finite (and sometimes infinite) state systems that relies on a symbolic representation of sets, typically as Binary Decision Diagrams (BDD's).

- **Verification** is used in industry for proving the correctness of complex digital circuits (control, arithmetic units, cache coherence), safety-critical software and embedded systems (traffic control, train systems, security protocols).

Verification Scenarios

Example

$$y = (x_1 + x_2) \cdot x_3$$



Comparison of specification and implementation

reference system → data structure

system under test → data structure

→ comparison

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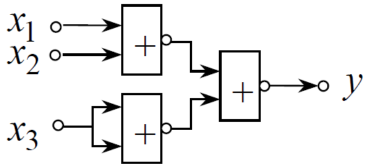

Comparison of specification and implementation

reference system → data structure → comparison

system under test → data structure → comparison

"The device can always be switched off."

Proving properties

property → fixed-point calculation

system under test → data structure → fixed-point calculation

Efficient state representation	- Set of states as Boolean function - Binary Decision Diagram representation
Computing reachability	- Leverage efficient state representation - Explore successor sets of states
Proving properties	- Temporal logic (CTL) - Encoding as reachability problem

This week

Efficient state representation

- Set of states as Boolean function
- Binary Decision Diagram representation

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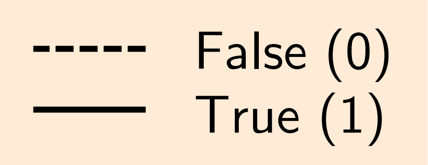
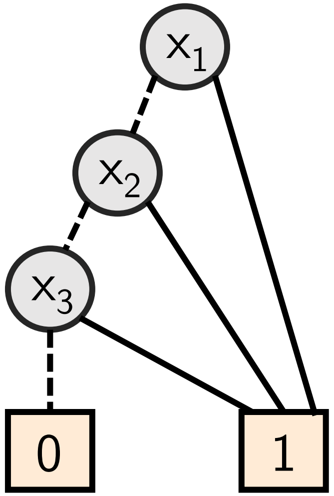
Binary Decision Diagrams (BDD)



- Concept
  - Data structure that allows to represent Boolean functions.
  - The representation is unique for a given ordering of variables. If the ordering of variables is fixed, we call it an ordered BDD (OBDD).

$$f = x_1 + x_2 + x_3$$



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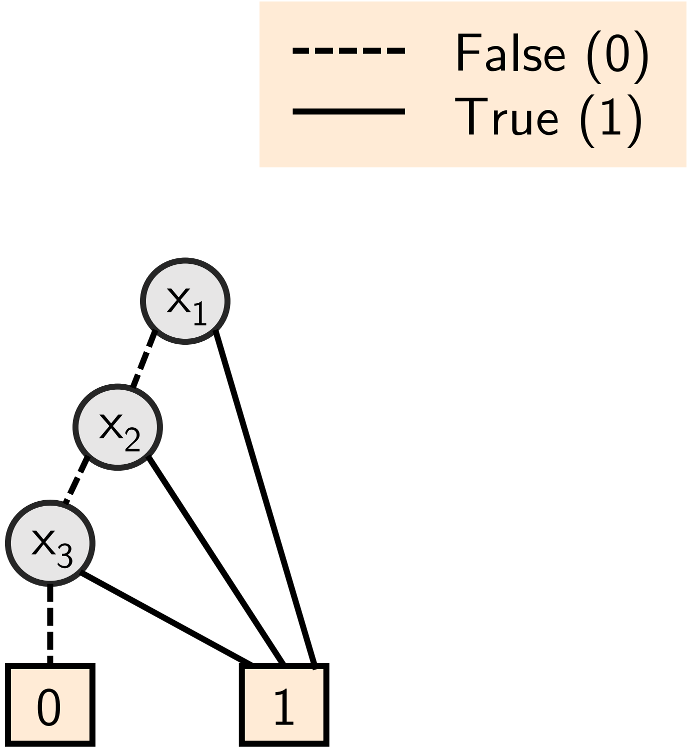

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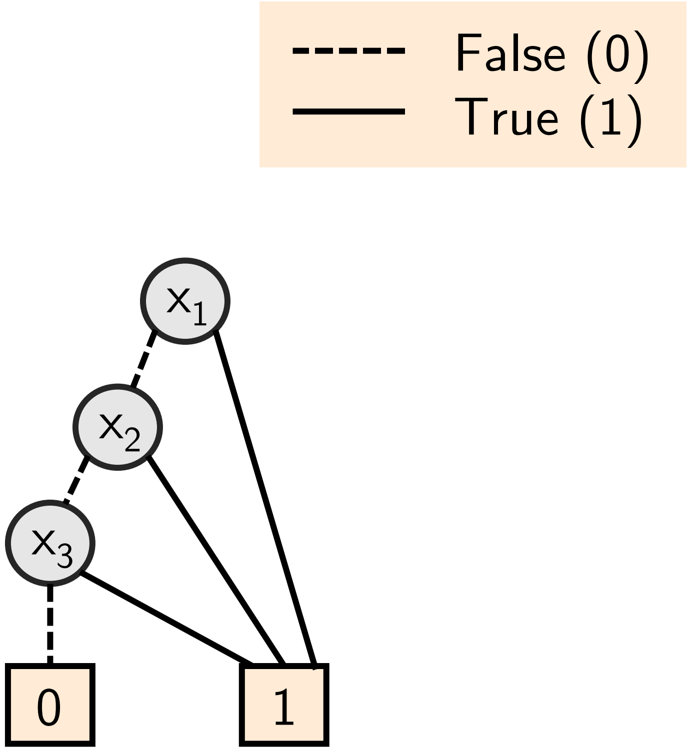

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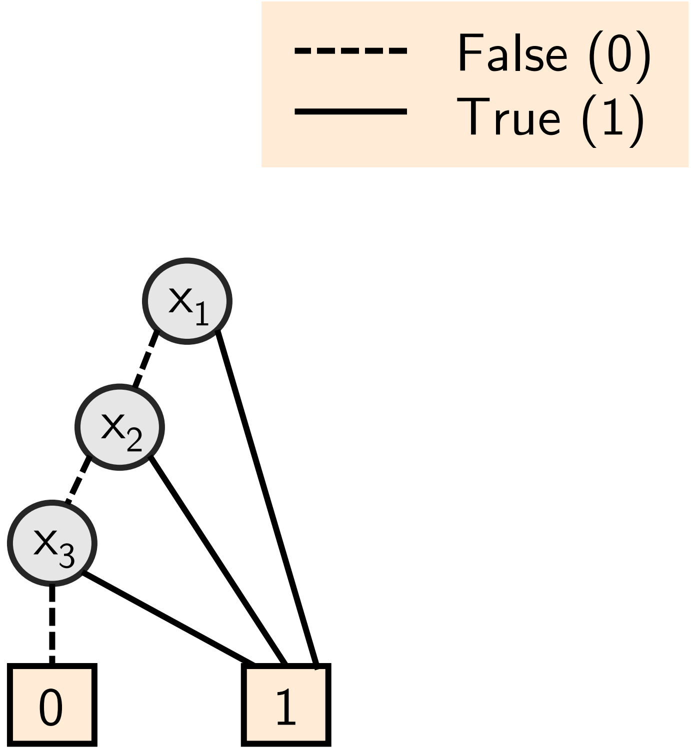

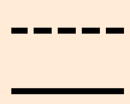

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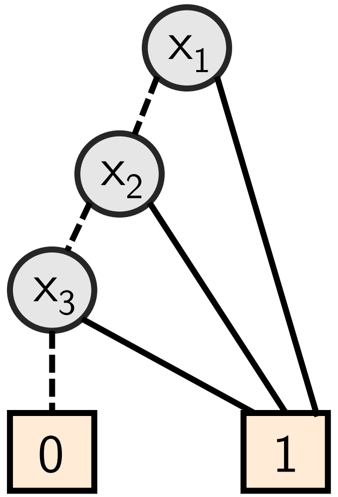

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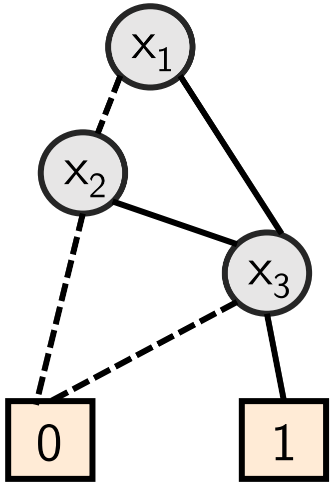

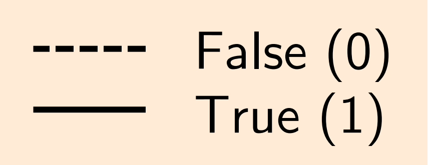

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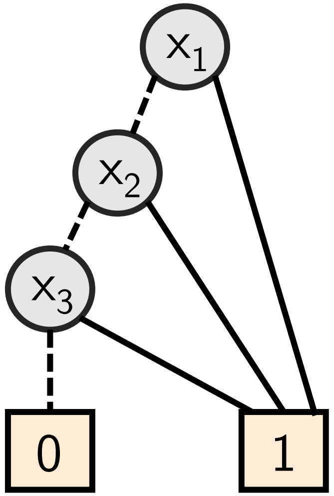

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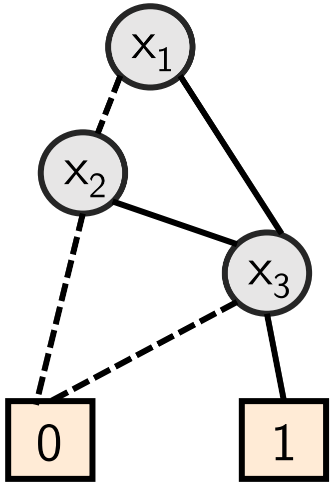

False (0)
True (1)

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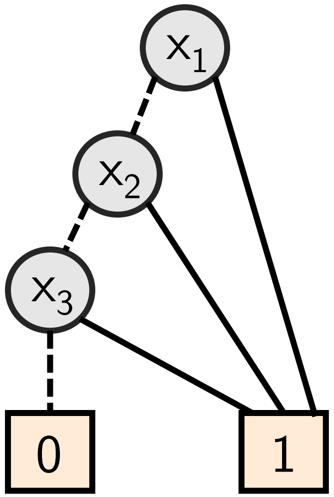

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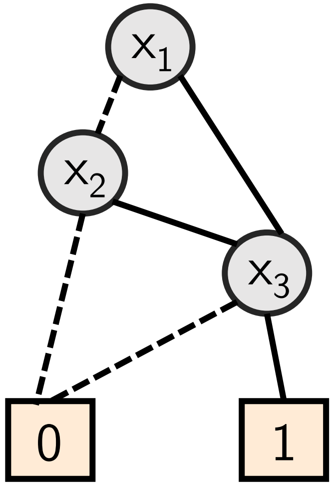

Basic concept of verification using BDDs

- BDDs represent Boolean functions.
- Therefore, they can be used to describe sets of states and transformation relations.
- Due to the unique representation of Boolean functions, *reduced ordered* BDDs (ROBDD) can be used to proof equivalence between Boolean functions or between sets of states.
- BDDs can easily and efficiently be manipulated.

Decomposition

Logic	Boolean	Binary
OR	$+$	$\vee$
AND	$\cdot$	$\wedge$
NOT	$\overline{X}$	$\neg$ or $\overline{X}$

BDDs are based on the Boole-Shannon-Decomposition:

$$f = \bar{x} \cdot f\Big|_{x=0} + x \cdot f\Big|_{x=1}$$

A Boolean function has two co-factors for each variable, one for each evaluation

- $f|_{x=0}$: remaining function for $x = 0$
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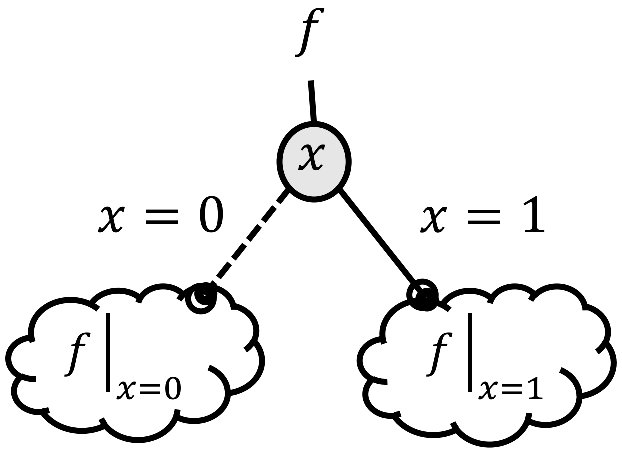

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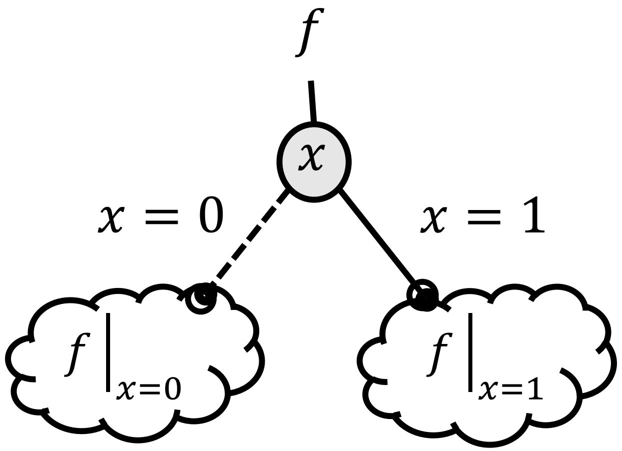

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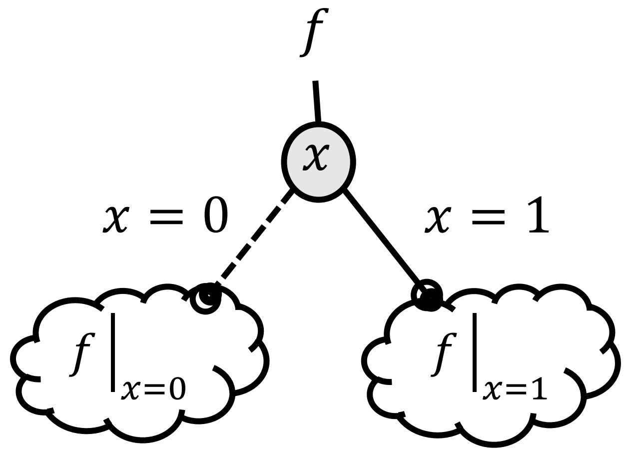
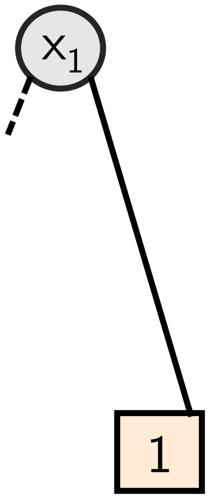


$$f = x_1 + x_2 + x_3$$
$$= x_1 \cdot \underbrace{1}_{f|_{x_1=1}} + \bar{x}_1 \cdot \underbrace{(x_2 + x_3)}_{f|_{x_1=0}}$$



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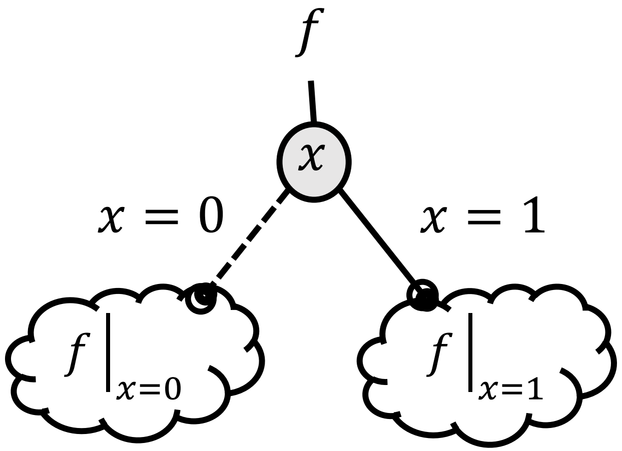

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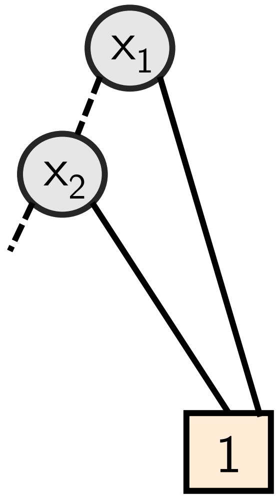

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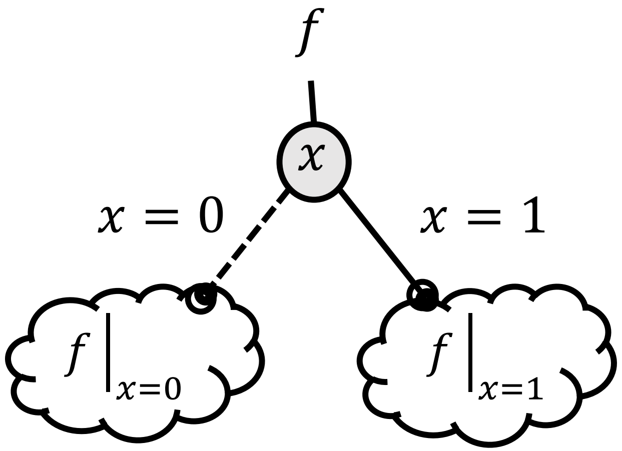

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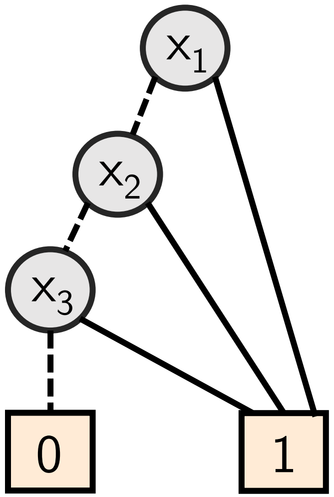

Boole-Shannon Decomposition Example

$f(a, b, c) = \bar{a} \cdot (b + c) + \bar{b} \cdot c$

Ordering: $a \ \rightarrow b \ \rightarrow c$

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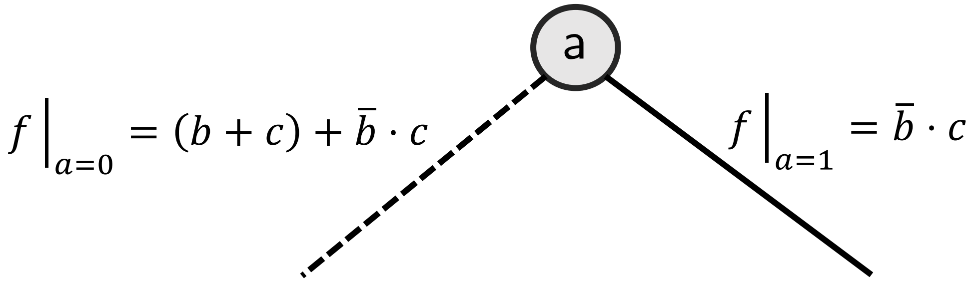

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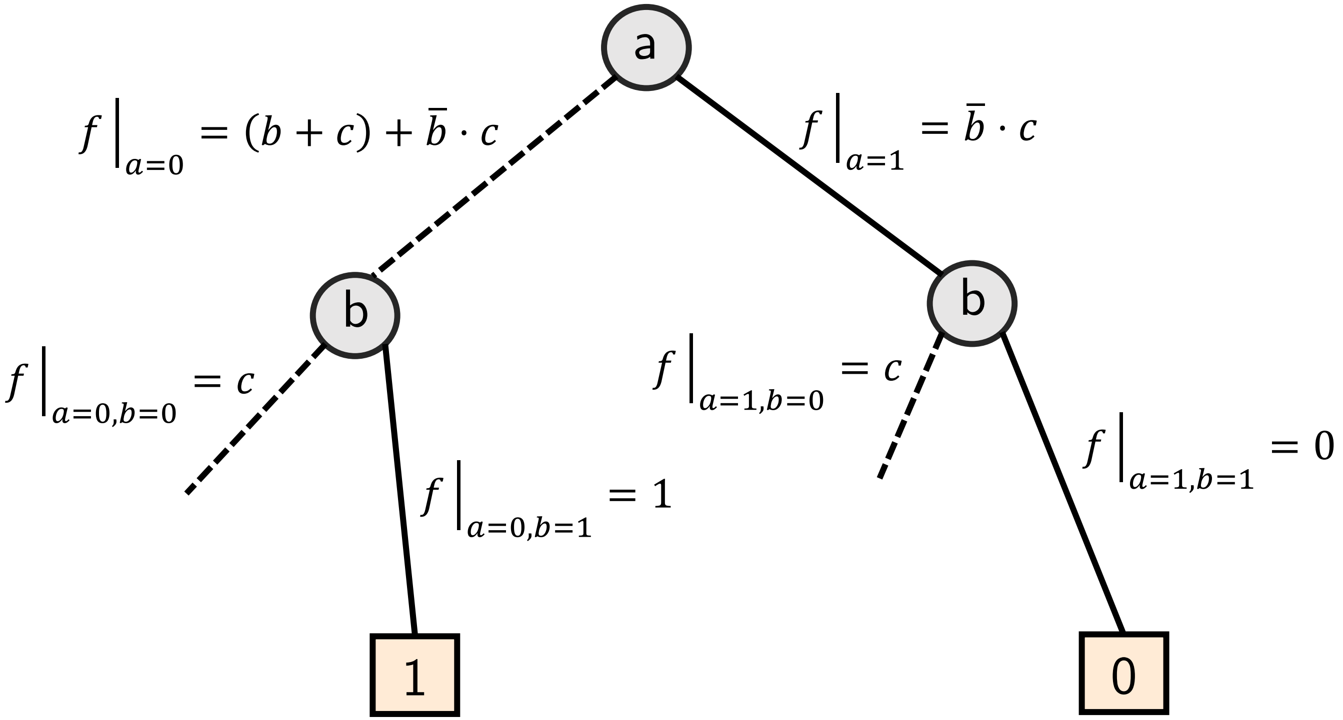

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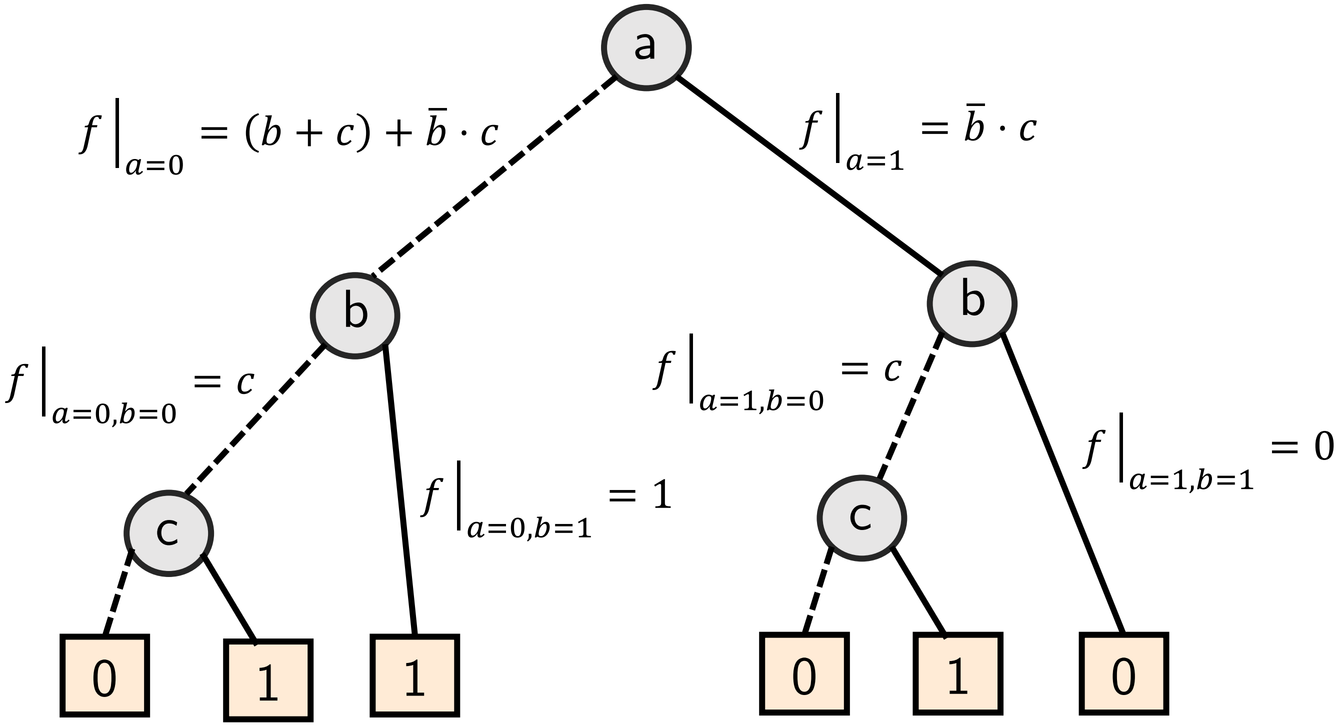

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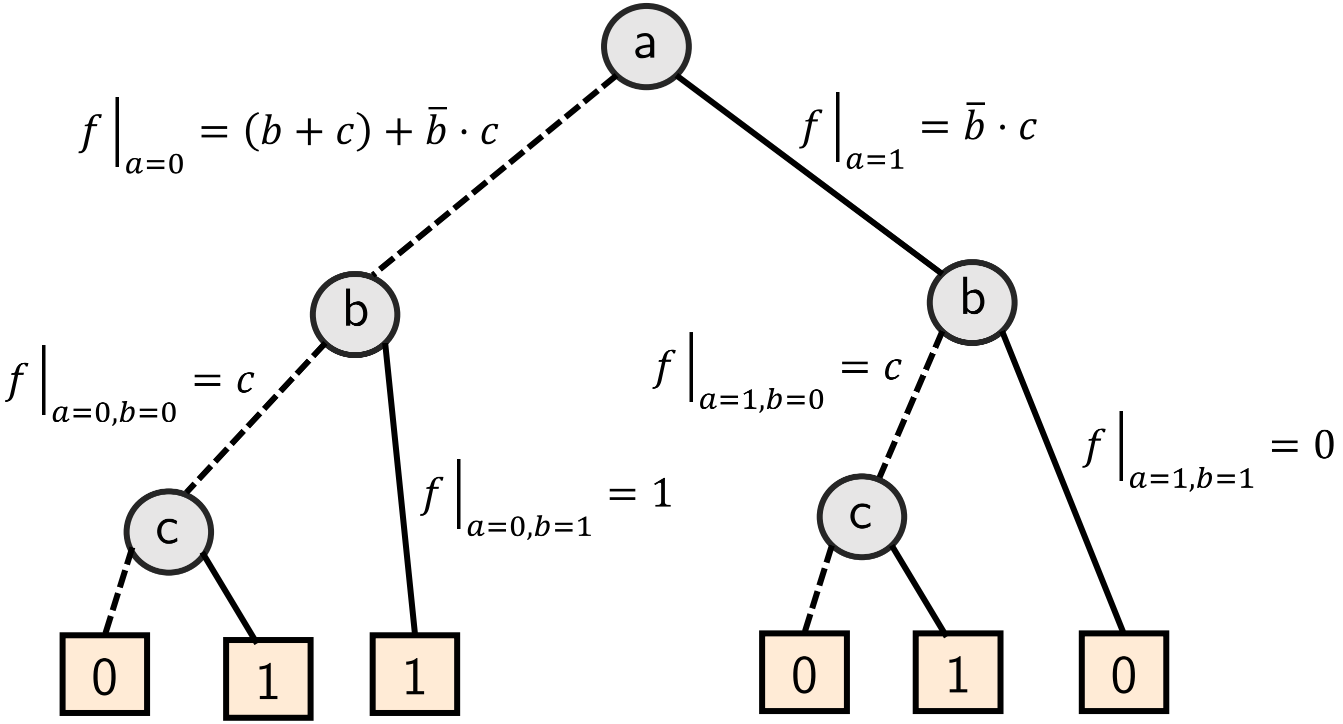
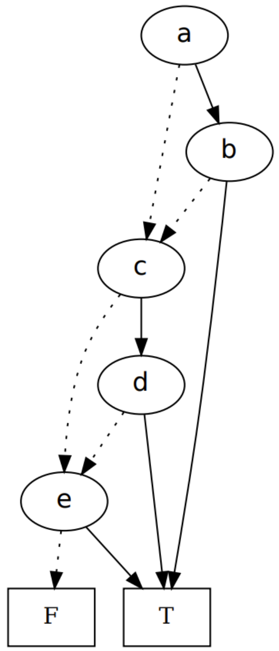


Does variable order matter?

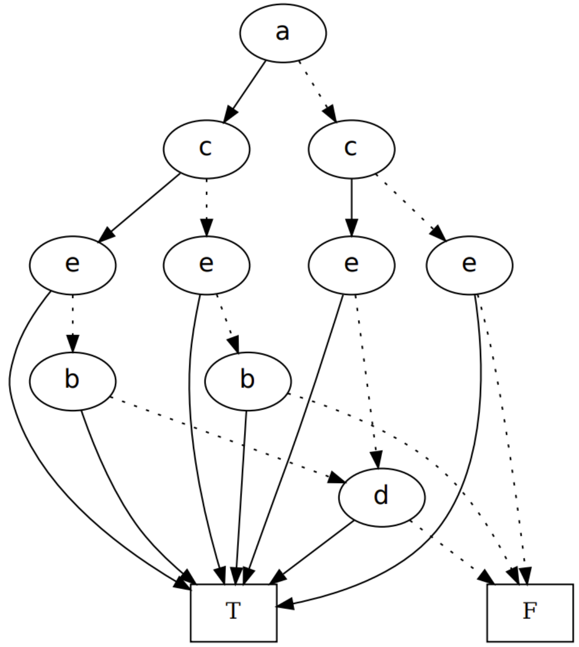
Variable Order

- If we fix the ordering of variables, BDDs are called OBBDs (Ordered Binary Decision Diagrams).
- The ordering is essential for the size of a BDD.

$$f = (a \cdot b) + (c \cdot d) + e$$



$$a \to b \to c \to d \to e$$



$$a \to c \to e \to b \to d$$

Calculating with BDDs

- ***SIMPLIFY***: Given BDD for f, determine simplified BDD for f.
  - Eliminate redundant nodes.
    - Merge equivalent leaves ($\boxed{0}$ and $\boxed{1}$)
    - Merge isomorphic nodes, i.e., nodes that represent the same Boolean function.
  - A BDD that can not be further simplified is called a reduced BDD.
    A reduced OBDD (also denoted as ROBDD) is a unique representation of a given Boolean function.

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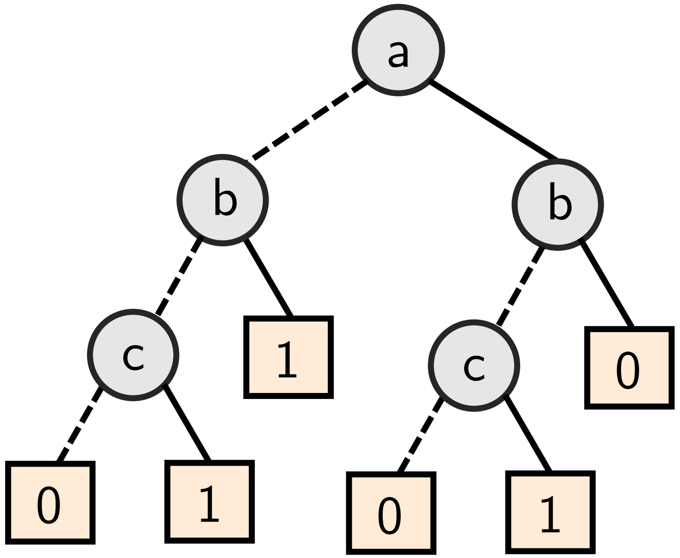

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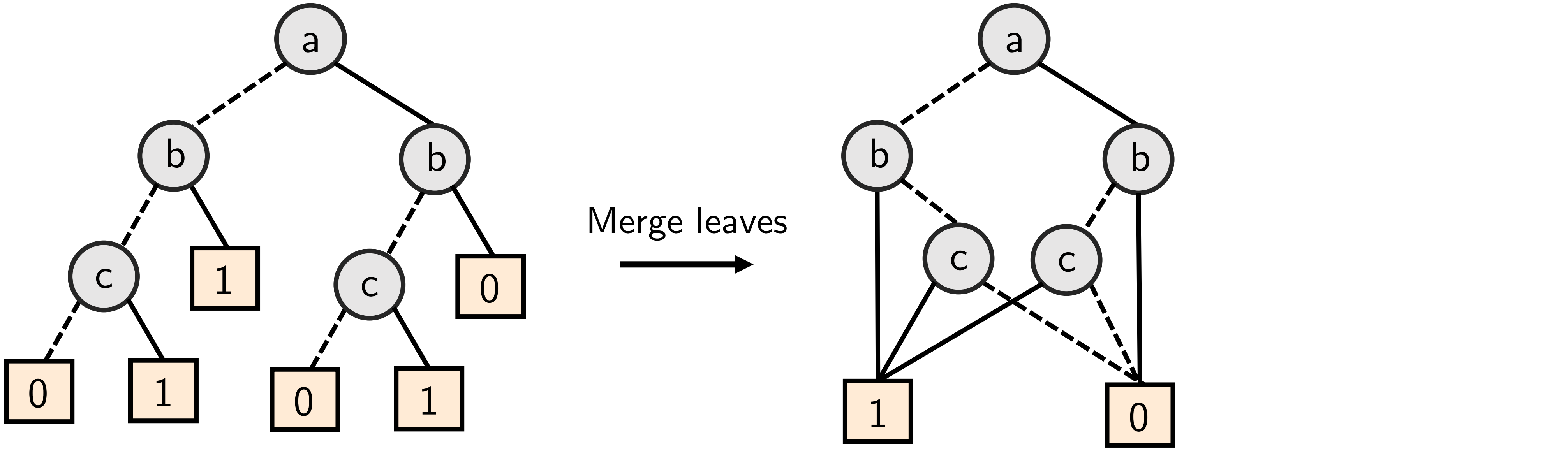

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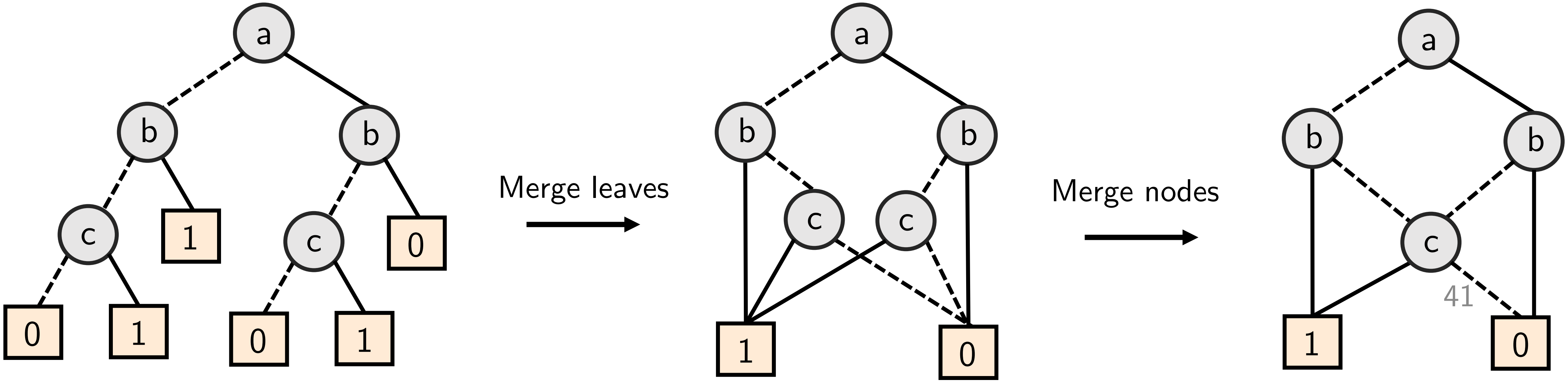
    A reduced OBDD (also denoted as ROBDD) is a unique representation of a given Boolean function.

$$f = \bar{a} \cdot (b + c) + \bar{b} \cdot c$$



Calculating with BDDs

- ***RESTRICT***: Given BDD for f, determine BDD for $f|_{x=k}$.
  - Delete all edges that represent $x = \bar{k}$;
  - For every pair of edges $(a - x\ ,\ \ x - b)$ include a new edge $(a - b)$ and remove the old ones;
  - Remove all nodes that represent x.

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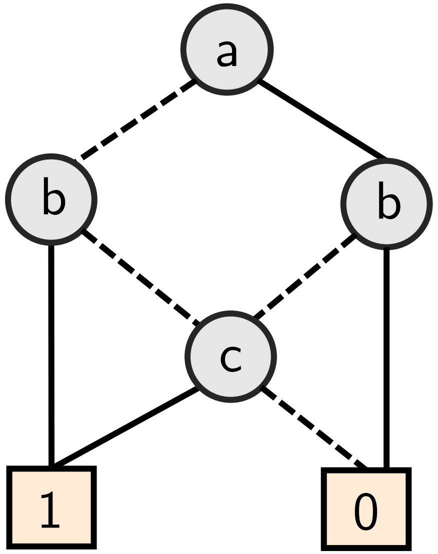

Restrict $f\Big|_{b=1}$ $\longrightarrow$

Calculating with BDDs

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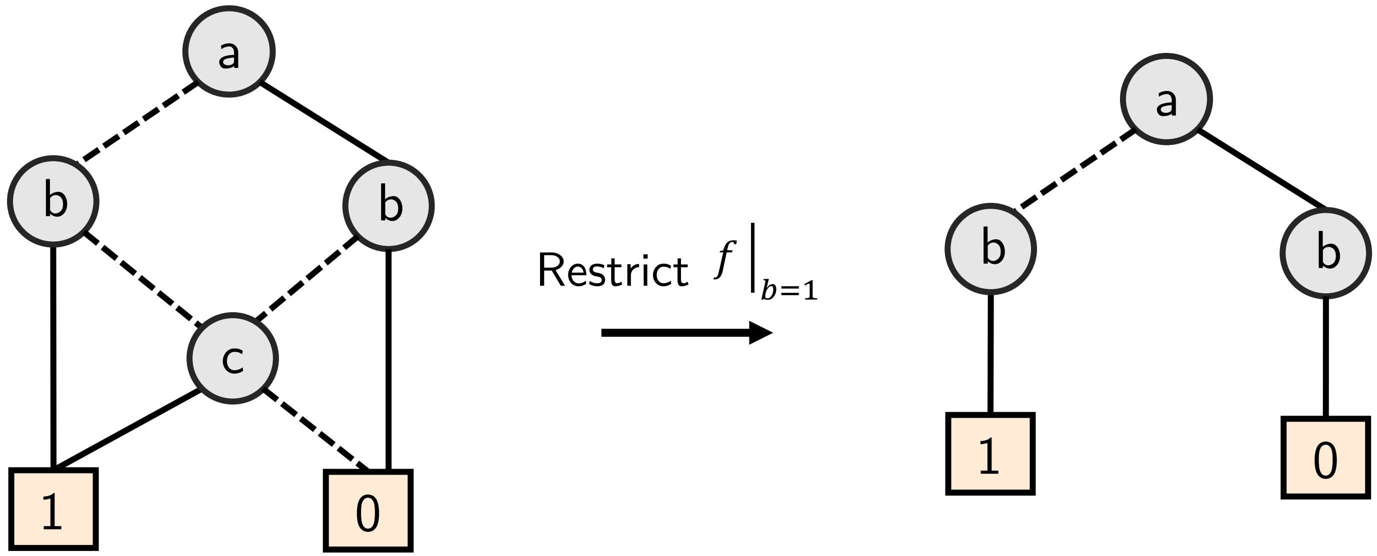

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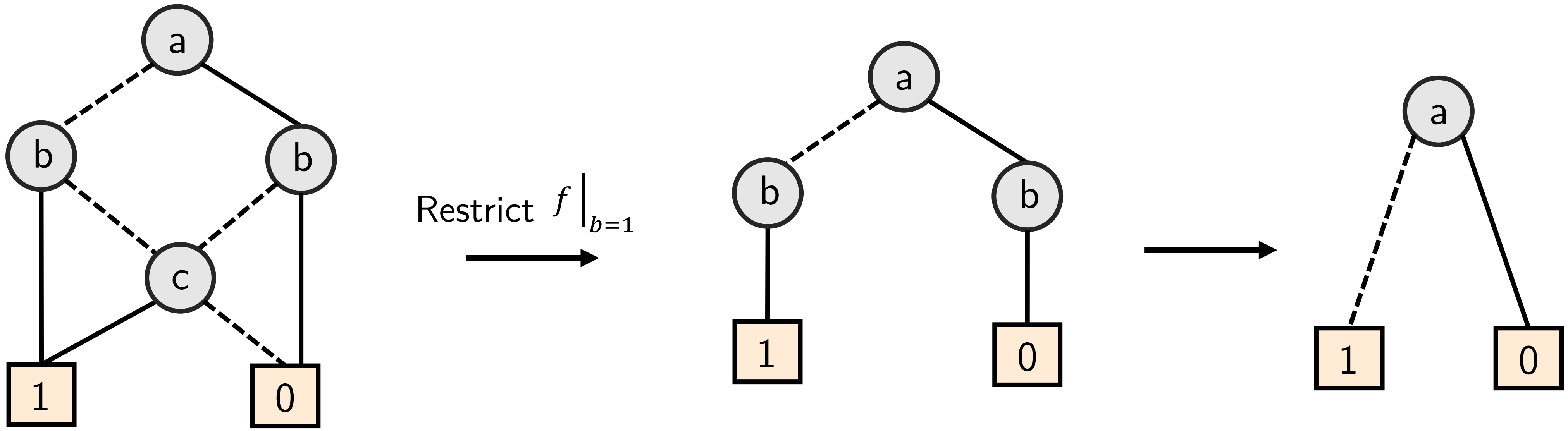
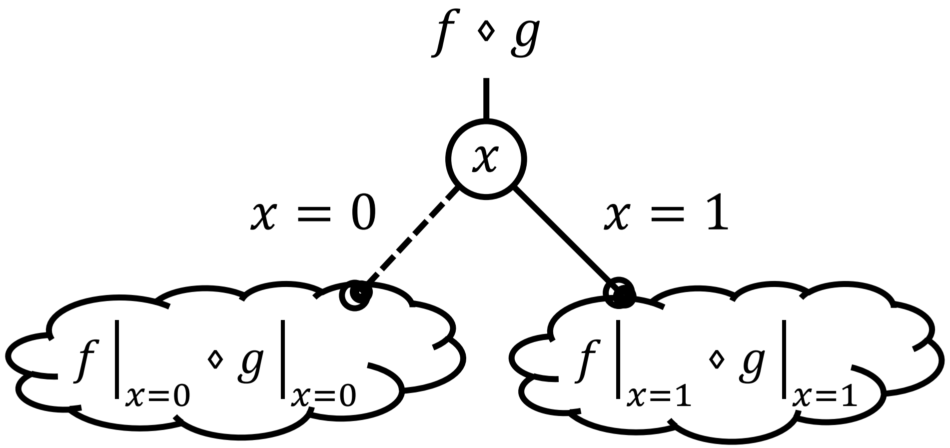


Calculating with BDDs

- ***APPLY***: Given BDDs for f and g, determine a BDD for $f \diamond g$ for some operation $\diamond$.
  - Combine the two BDDs recursively based on the following relation:

$$f \diamond g = \overline{x} \cdot (f\mid_{x=0} \diamond\, g\mid_{x=0}) + x \cdot (f\mid_{x=1} \diamond\, g\mid_{x=1})$$



- Boolean functions can be converted to BDDs step by step using ***APPLY***.

Calculating with BDDs

- Quantifiers are constructed by ***APPLY*** and ***RESTRICT***:

$$(\exists x : f) \quad \Leftrightarrow \quad (f \mid_{x=0} + f \mid_{x=1})$$

$$(\forall x : f) \quad \Leftrightarrow \quad (f \mid_{x=0} \cdot f \mid_{x=1})$$

$$(\exists x_1, x_2 : f) \quad \Leftrightarrow \quad (\exists x_1 \ (\exists x_2 : f))$$

$$(\forall x_1, x_2 : f) \quad \Leftrightarrow \quad (\forall x_1 \ (\forall x_2 : f))$$

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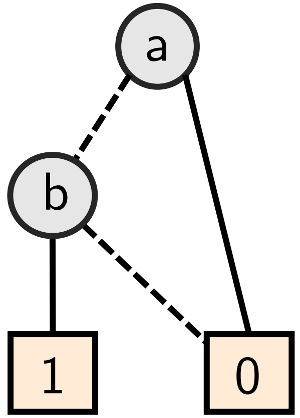

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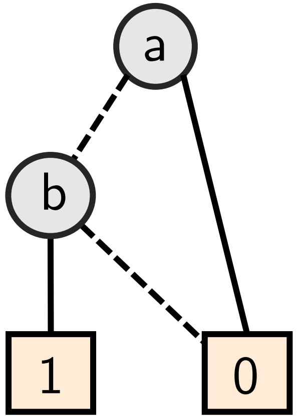
$g(a) = \exists b\colon f(a, b)$



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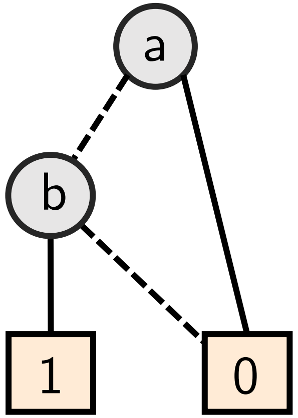

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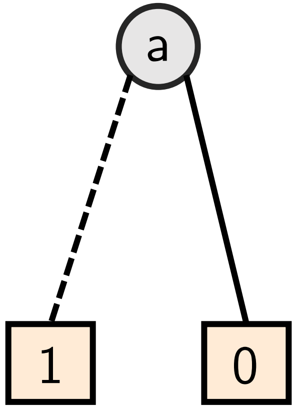

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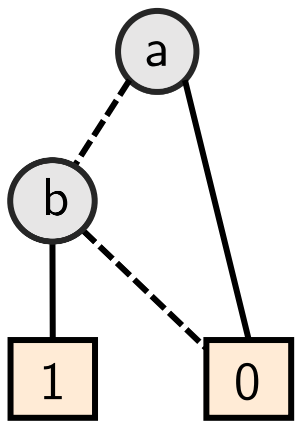

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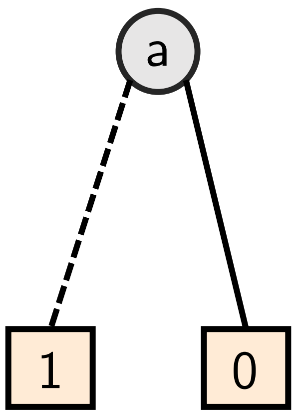

$h(a) = \forall b\colon f(a, b)$

Calculating with BDDs

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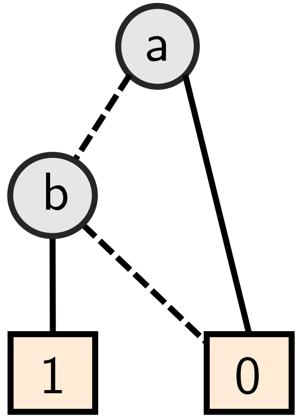

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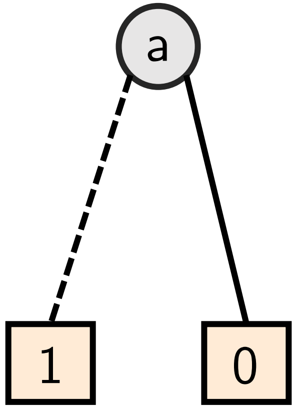
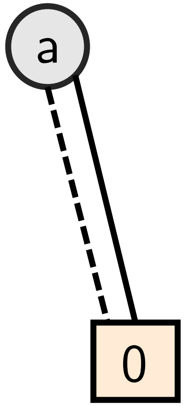
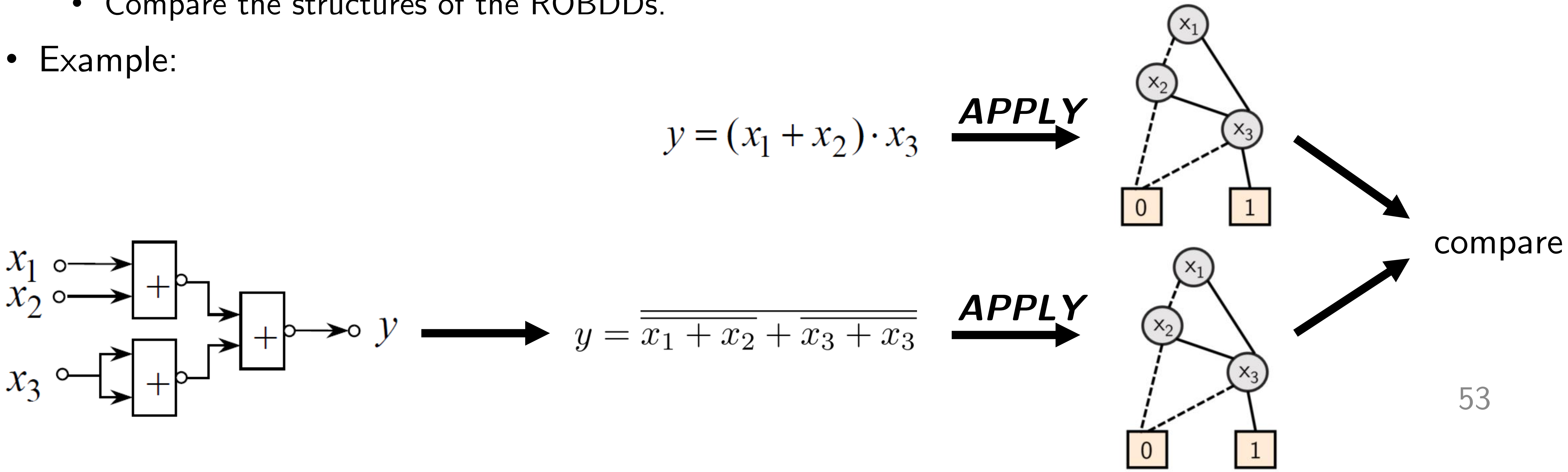


$h(a) = \forall b \colon f(a, b) = 0$



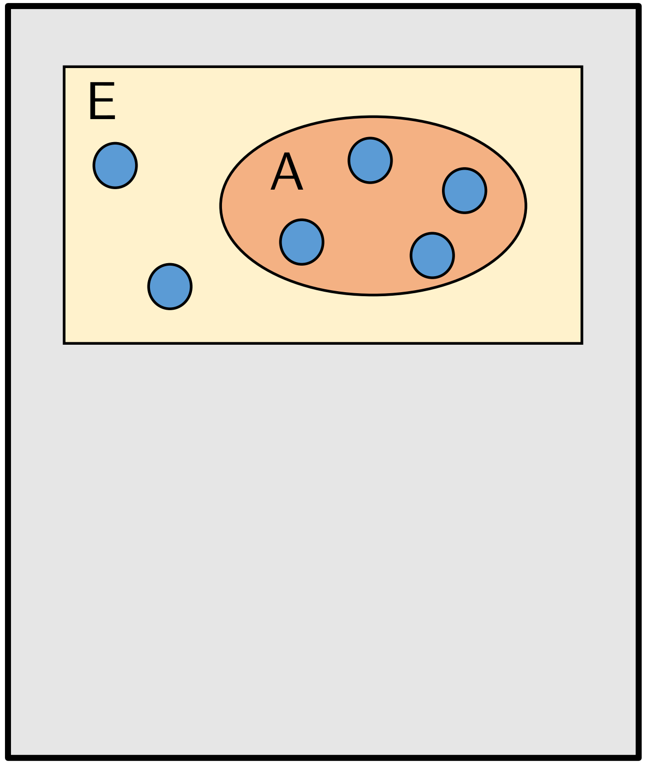
Comparison using BDDs

- Boolean (combinatorial) circuits: Compare specification and implementation, or compare two implementations.
- Method:
  - Representation of the two systems in ROBDDs, e.g., by applying the ***APPLY*** operator repeatedly.
  - Compare the structures of the ROBDDs.
- Example:



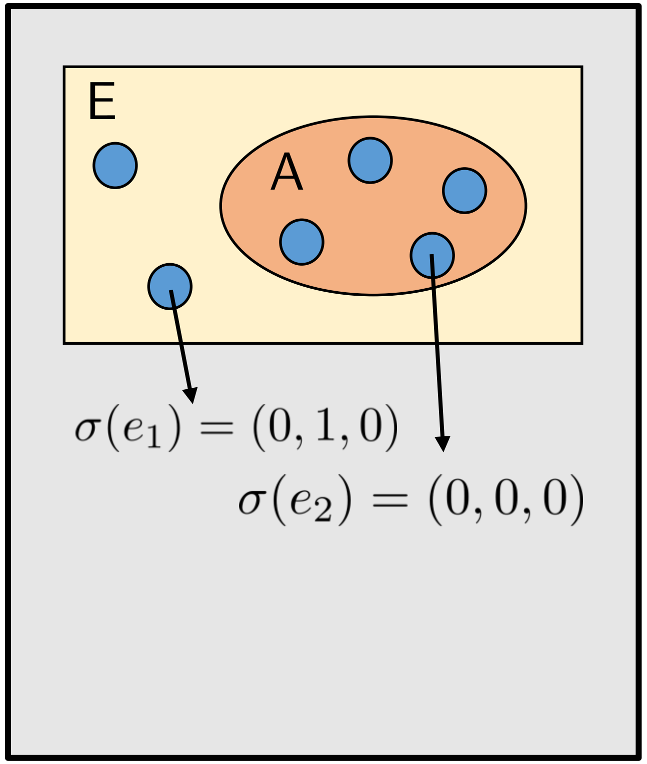
Sets and Relations

- Representation of a subset $A \subseteq E$:



Sets and Relations

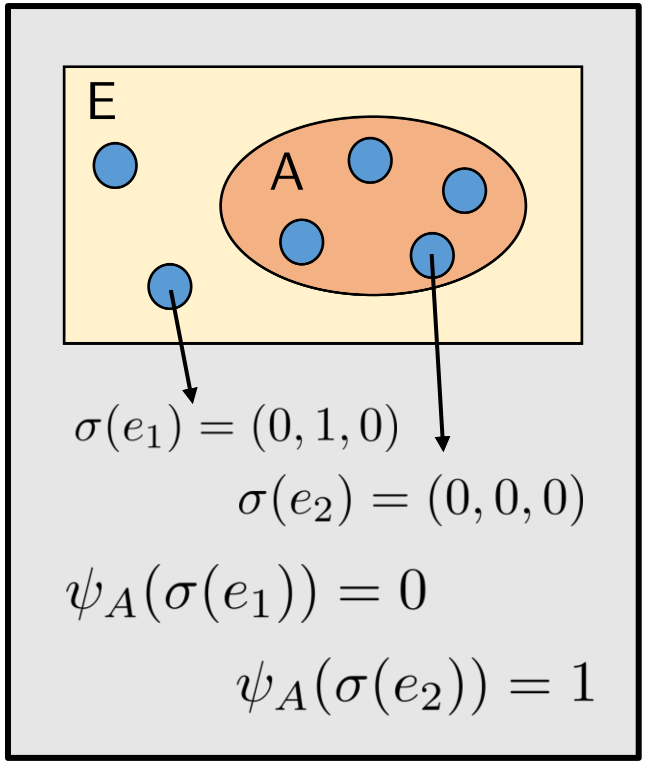
- Representation of a subset $A \subseteq E$:
  - Binary encoding $\sigma(e)$ of all elements $e \in E$



Sets and Relations

- Representation of a subset $A \subseteq E$:
  - Binary encoding $\sigma(e)$ of all elements $e \in E$
  - Subset A is represented by $a \in A \Leftrightarrow \psi_A(\sigma(a))$

characteristic function of subset A



Sets and Relations

- Representation of a subset $A \subseteq E$:
  - Binary encoding $\sigma(e)$ of all elements $e \in E$
  - Subset A is represented by $a \in A \Leftrightarrow \psi_A(\sigma(a))$
  - Stepwise construction of the BDD corresponding to some subsets.

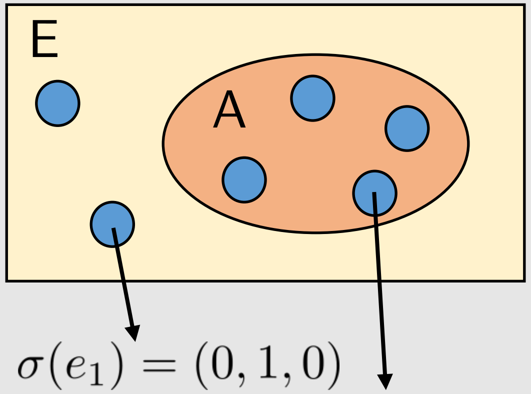
characteristic function of subset A

$$c \in A \cap B \quad \Leftrightarrow \quad \psi_A(\sigma(c)) \cdot \psi_B(\sigma(c))$$

$$c \in A \cup B \quad \Leftrightarrow \quad \psi_A(\sigma(c)) + \psi_B(\sigma(c))$$

$$c \in A \setminus B \quad \Leftrightarrow \quad \psi_A(\sigma(c)) \cdot \overline{\psi_B(\sigma(c))}$$

$$c \in E \setminus A \quad \Leftrightarrow \quad \overline{\psi_A(\sigma(c))}$$



$$\sigma(e_1) = (0,1,0)$$

$$\sigma(e_2) = (0,0,0)$$

$$\psi_A(\sigma(e_1)) = 0$$

$$\psi_A(\sigma(e_2)) = 1$$

Sets and Relations

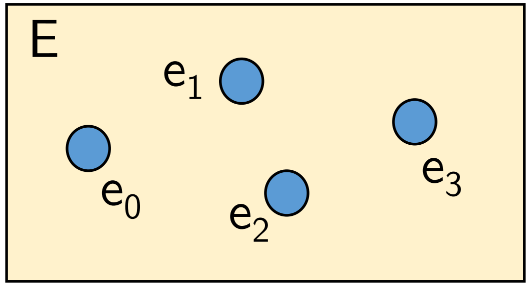
- Example:

$$\forall e \in E: \ \sigma(e) = (x_1, x_0)$$

$$\sigma(e_0) = (0,0) \quad \sigma(e_1) = (0,1) \quad \sigma(e_2) = (1,0) \quad \sigma(e_3) = (1,1)$$

$$\psi_A = x_0 \oplus x_1$$

$A =$?



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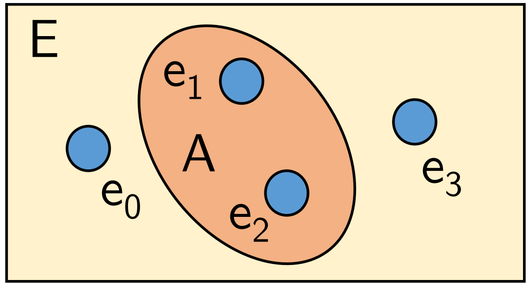
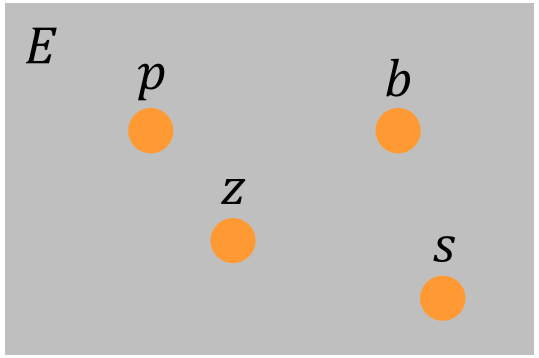
$$\psi_A = x_0 \oplus x_1$$

$$A = \{e_1, e_2\}$$



Sets and Relations

$\boldsymbol{\sigma(e)}$	$\mathbf{x_1}$	$\mathbf{x_0}$
Zürich	0	0
Sydney	0	1
Beijing	1	0
Paris	1	1



Capitals? $\psi_A(x_1, x_0) = ?$

European cities? $\psi_B(x_1, x_0) = ?$

European capitals? $\psi_C(x_1, x_0) = ?$

Sets and Relations

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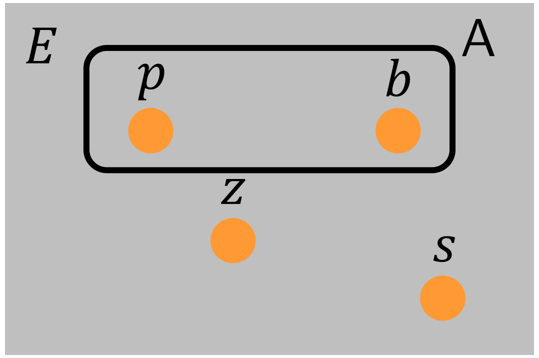

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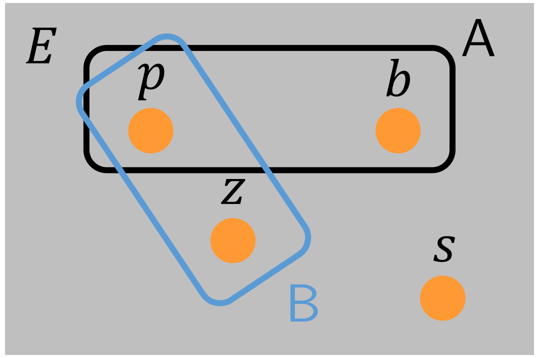

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Sets and Relations

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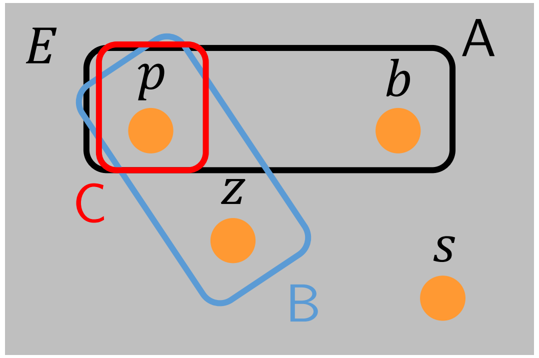

Capitals? $\psi_A(x_1, x_0) = ?$ $\psi_A(x_1, x_0) = x_1$

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European capitals? $\psi_c(x_1, x_0) = ?$ $C = A \cap B$ $\psi_c(x_1, x_0) = x_0 \cdot x_1$

Selecting a "good" encoding is both important and difficult

For a state space encoded with N bits

Represent up to 2^N states

In previous example

Subset A of all capitals is represented by $\psi_A = x_1$

- No need to iterate through all capitals to verify that some property holds (e.g. "All capitals have a parliament.")
- We can use the (compact) representation of the set.

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But...

Selecting a good encoding —Representing state efficiently is difficult in practice.

- It is one challenge of ML: How to efficiently encode the inputs?

Efficient state representation

- Set of states as Boolean function
- Binary Decision Diagram representation

Computing reachability

- Leverage efficient state representation
- Explore successor sets of states

Proving properties

- Temporal logic (CTL)
- Encoding as reachability problem

Sets and Relations using BDDs

- Representation of a relation $R \subseteq A \times B$
    - Binary encoding $\sigma(a)$, $\sigma(b)$ of all elements $a \in A$, $b \in B$
    - Representation of R

$$(a, b) \in R \Leftrightarrow \psi_R(\sigma(a), \sigma(b))$$

characteristic function of the relation R

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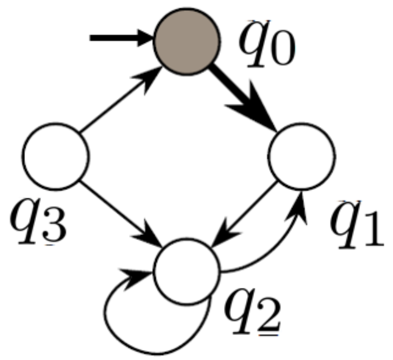
- Example:



$$\psi_\delta(\sigma(q), \sigma(q')) = \psi_\delta(q, q')$$

To simplify notation

$$q \xrightarrow{\delta} q'$$

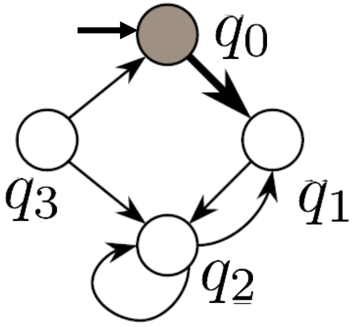
describe state transitions
return 1 if there is a transition
$q \rightarrow q'$, 0 otherwise

$\psi_\delta(q_0, q_1) = 1$
$\psi_\delta(q_0, q_3) = 0$

Reachability of States

- Problem: Is a state $q \in Q$ reachable by a sequence of state transitions?
- Method:
  - Represent set of states and the transformation relation as ROBDDs.
  - Use these representations to transform from one set of states to another. Set Q_i corresponds to the set of states reachable after i transitions.
  - Iterate the transformation until a fixed-point is reached, i.e., until the set of states does not change anymore (steady-state).
- Example:

$Q_0 = \{q_0\}$



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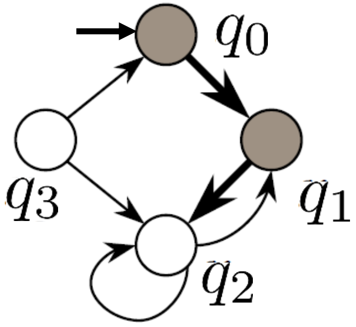
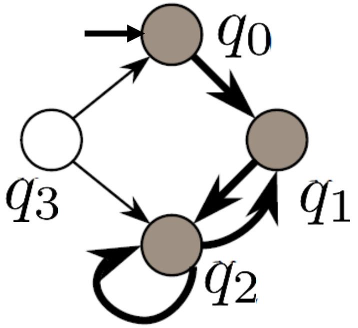
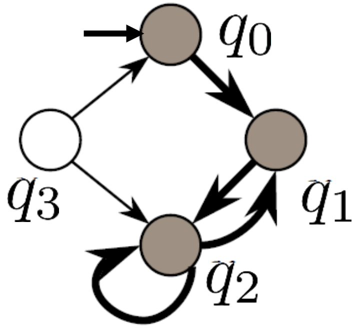


$Q_2 = \{q_0, q_1, q_2\}$



$Q_3 = \{q_0, q_1, q_2\}$



Drawing state-diagrams is not feasible in general.

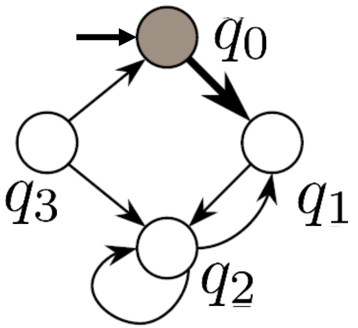
Drawing state-diagrams is not feasible in general.

1. Work with sets of states
2. Use characteristic functions to represent sets of states
3. Use ROBDDs to encode characteristic functions

Reachability of States

- Transformation of sets of states:
  - Determine the set of all direct successor states of a given set of states Q by means of the transformation function δ:

Set of successor states: $Q' = Suc(Q, \delta) = \{q' \mid \exists q : \underbrace{\psi_Q(q)}_{\text{Characteristic function of current state set } Q} \cdot \underbrace{\psi_\delta(q, q')}_{\text{Transition function } q \rightarrow q'}\}$



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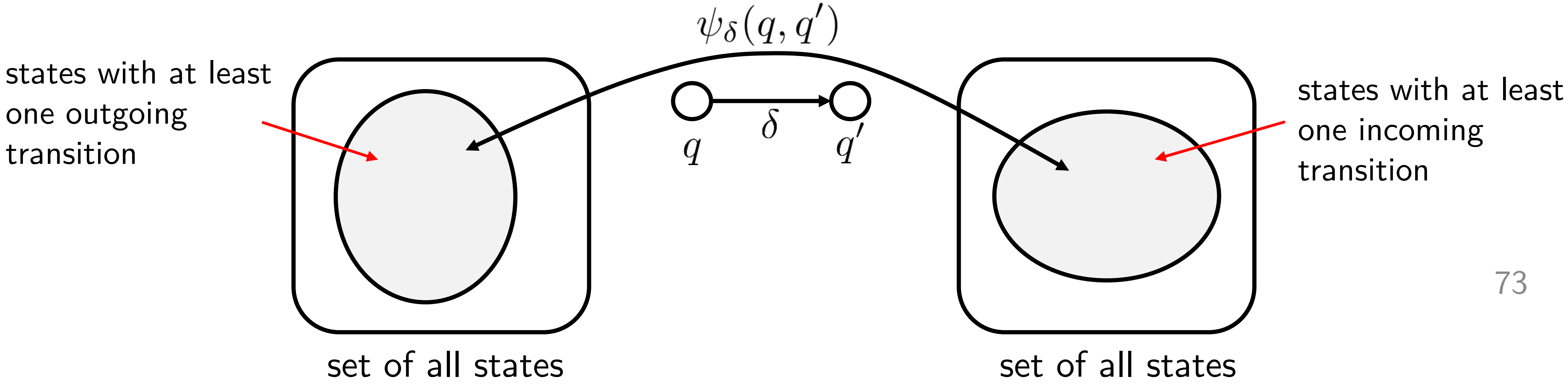

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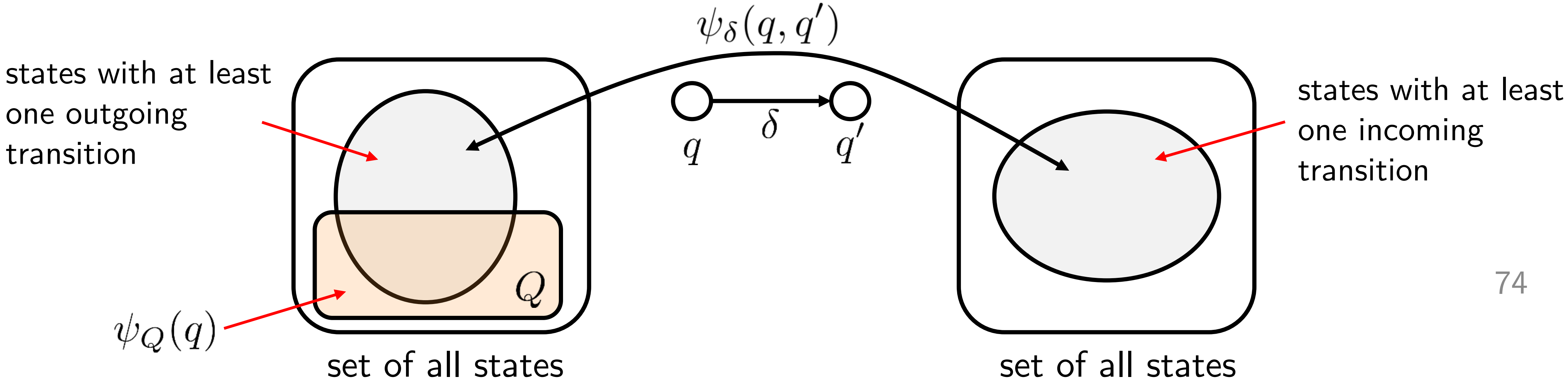

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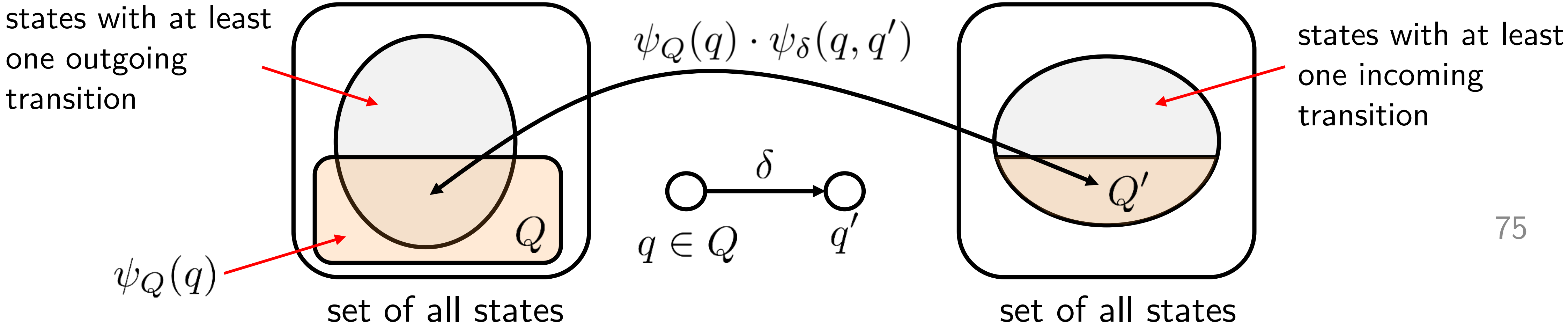

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$$h(q, q') = \psi_Q(q) \cdot \psi_\delta(q, q')$$

$$\psi_{Q'}(q') = (\exists q : h(q, q'))$$

Reachability of States

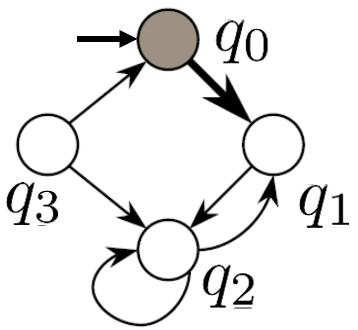
- Fixed-point iteration
  - Start with the initial state, then determine the set of states that can be reached in one or more steps.

$$Q_0 = \{q_0\}$$

$$\boxed{Q_{i+1}} = Q_i \cup Suc(Q_i, \delta) \qquad \text{until } Q_{i+1} = Q_i$$

Characteristic function of next set of reached states

$$\psi_{Q_{i+1}}(q') = \underbrace{\psi_{Q_i}(q')}_{q' \text{ is already in } Q_i} + \underbrace{(\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))}_{\text{There is a state } q \text{ in } Q_i \text{ with transition } q \to q'}$$



$$Q_0 = \{q_0\}$$

$$Q' = Suc(Q_0, \delta) = \{q_1\}$$

$$Q_1 = Q_0 \cup Suc(Q_0, \delta) = \{q_0, q_1\}$$

Reachability of States

- Fixed-point iteration
  - Start with the initial state, then determine the set of states that can be reached in one or more steps.

$$Q_0 = \{q_0\}$$

$$Q_{i+1} = Q_i \cup Suc(Q_i, \delta) \qquad \text{until } Q_{i+1} = Q_i$$

Characteristic function of next set of reached states

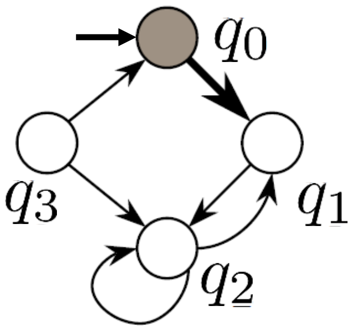
$$\psi_{Q_{i+1}}(q') = \underbrace{\psi_{Q_i}(q')}_{q' \text{ is already in } Q_i} + \underbrace{(\,\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))}_{\text{There is a state } q \text{ in } Q_i \text{ with transition } q \to q'}$$

- Due to the finite number of states, the fixed-point exists and is reached in a finite number of steps (at most the diameter of the state diagram).
- Determine whether the fixed-point is reached or not can be done by comparing the ROBDDs of the current set of reachable states.

Reachability of States - Example

$\sigma(q)$	x_1	x_0
q_0	0	0
q_1	0	1
q_2	1	0
q_3	1	1

State encoding
$(x_1, x_0) = \sigma(q)$



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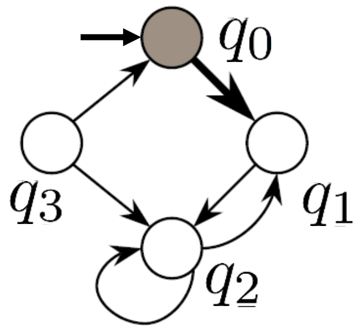
Transition relation encoding $\psi_\delta(q, q')$

As a Boolean function

$\psi_\delta(q, q') = \overline{x_0{}'} \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1{}'}$

entries where
$\psi_\delta(q, q') = 1$ only

x_1	x_0	x_1'	x_0'	
0	0	0	1	$q_0 \rightarrow q_1$
0	1	1	0	
1	0	0	1	
1	0	1	0	$q_2 \rightarrow q_2$
1	1	1	0	
1	1	0	0	



Reachability of States - Example

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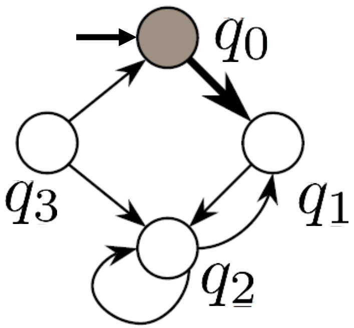
Compute reachable states:

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\, \exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q,q'))$$



Reachability of States - Example

$\boldsymbol{\sigma(q)}$	**x$_1$**	**x$_0$**
q_0	0	0
q_1	0	1
q_2	1	0
q_3	1	1

State encoding

$$(x_1, x_0) = \sigma(q)$$

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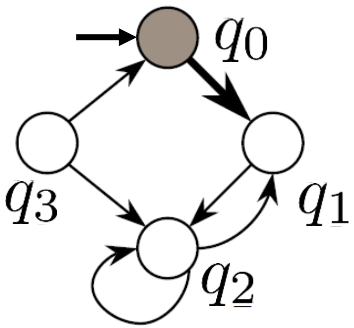
$$Q_0 = \{q_0\}$$

$$\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}$$



Reachability of States - Example

$\boldsymbol{\sigma(q)}$	**x₁**	**x₀**
q_0	0	0
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State encoding

$(x_1, x_0) = \sigma(q)$

Transition relation encoding $\psi_\delta(q, q')$

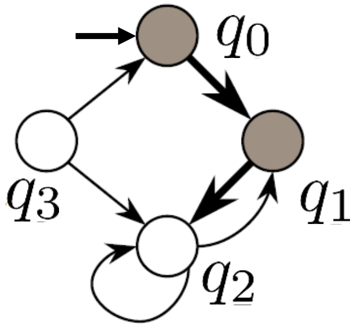
As a Boolean function

$\psi_\delta(q,q') = \overline{x_0}' \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1}'$

Compute reachable states:

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q,q'))$$

$$Q_1 = Q_0 \cup \{q_1\} = \{q_0, q_1\}$$



$$\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}$$

$$\psi_{Q_1}(q') = \overline{x_1'} \cdot \overline{x_0'} + (\exists q : \overline{x_1} \cdot \overline{x_0} \cdot \psi_\delta(q,q'))$$

Reachability of States - Example

$\boldsymbol{\sigma(q)}$	**x$_1$**	**x$_0$**
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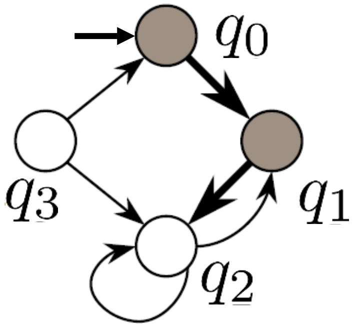
Compute reachable states:

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q,q'))$$

$$Q_1 = Q_0 \cup \{q_1\}$$
$$= \{q_0, q_1\}$$



$$\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}$$

$$\psi_{Q_1}(q') = \overline{x_1'} \cdot \overline{x_0'} + \underbrace{(\exists q : \overline{x_1} \cdot \overline{x_0} \cdot \psi_\delta(q,q'))}_{q_0:\ x_0=0,\ x_1=0}$$

$$= \overline{x_1'} \cdot \overline{x_0'} + \overline{x_1'} \cdot x_0' = \overline{x_1'}$$

From BDDs and quantifiers:

$$\exists x: f = f\big|_{x=0} + f\big|_{x=1}$$

The only non-zero term is for x_0=0, x_1=0 (see next slide)

Reachability of States – Example (BDD Calculation)

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\,\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q,q'))$$

$$\psi_{Q_0}(q) = \overline{x_1} \cdot \overline{x_0}$$

Eq$_1$: $$\psi_{Q_1}(q') = \overline{x_1'} \cdot \overline{x_0'} + \underbrace{(\,\exists q : \overline{x_1} \cdot \overline{x_0} \cdot \psi_\delta(q,q'))}_{\exists q : f \,\to\, \exists x_1 \exists x_0 : f}$$

From BDDs and quantifiers:

$$(\exists x_1, x_2 : f) \quad \Leftrightarrow \quad (\exists x_1\ (\exists x_2 : f))$$

$$\exists x : f = f\big|_{x=0} + f\big|_{x=1}$$

$\exists x_0 : f$

$$f\big|_{x_0=1} = \overline{x_1} \cdot 0 \cdot (\overline{x_0{}'} \cdot (1 \cdot (x_1 + x_1') + x_1 \cdot x_1') + 0 \cdot x_0' \cdot \overline{x_1{}'}) = 0$$

$$f|_{x_0=0} = \overline{x_1} \cdot 1 \cdot (\overline{x_0{}'} \cdot (0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + 1 \cdot x_0' \cdot \overline{x_1{}'}) = \overline{x_1} \cdot (\overline{x_0{}'} \cdot (x_1 \cdot x_1') + x_0' \cdot \overline{x_1{}'})$$

$\exists x_1 : f\big|_{x_0=0}$

$$f|_{x_0=0, x_1=1} = 0 \cdot \ (\overline{x_0{}'} \cdot (1 \cdot x_1') + x_0' \cdot \overline{x_1{}'}) = 0$$

$$f|_{x_0=0, x_1=0} = 1 \cdot \ (\overline{x_0{}'} \cdot (0 \cdot x_1') + x_0' \cdot \overline{x_1{}'}) = x_0' \cdot \overline{x_1{}'}$$

$\exists x_1 \exists x_0 : f = \ x_0' \cdot \overline{x_1{}'}$

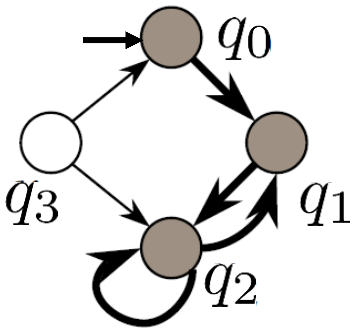
Plug into Eq$_1$ to compute $\psi_{Q_1}(q')$

Reachability of States - Example

$\sigma(q)$	x_1	x_0
q_0	0	0
q_1	0	1
q_2	1	0
q_3	1	1

State encoding
$(x_1, x_0) = \sigma(q)$

$$Q_2 = Q_1 \cup \{q_1, q_2\}$$
$$= \{q_0, q_1, q_2\}$$



Transition relation encoding $\psi_\delta(q, q')$

As a Boolean function

$$\psi_\delta(q, q') = \overline{x_0}' \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1}'$$

Compute reachable states:

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\, \exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))$$

$$\psi_{Q_1}(q') = \overline{x_1'}$$

$$\psi_{Q_2}(q') = \overline{x_1'} + (\, \exists q : \overline{x_1} \cdot \psi_\delta(q, q'))$$

Reachability of States - Example

$\boldsymbol{\sigma(q)}$	**x1**	**x0**
q_0	0	0
q_1	0	1
q_2	1	0
q_3	1	1

State encoding

$(x_1, x_0) = \sigma(q)$

Transition relation encoding $\psi_\delta(q, q')$

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$\psi_\delta(q,q') = \overline{x_0}' \cdot (x_0 \cdot (x_1 + x_1') + x_1 \cdot x_1') + \overline{x_0} \cdot x_0' \cdot \overline{x_1}'$

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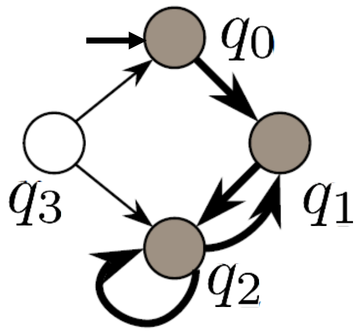
$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\,\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q,q'))$$

$Q_2 = Q_1 \cup \{q_1, q_2\}$
$= \{q_0, q_1, q_2\}$



$$\psi_{Q_1}(q') = \overline{x_1'}$$

$$\psi_{Q_2}(q') = \overline{x_1'} + \underbrace{(\,\exists q : \overline{x_1} \cdot \psi_\delta(q,q'))}$$

q_0: x_0=0, x_1=0

q_1: x_0=1, x_1=0

$$= \overline{x_1'} + \overline{x_1'} \cdot x_0' + \overline{x_1'} \cdot \overline{x_0'} = \overline{x_1'} + \overline{x_0'}$$

Reachability of States - Example

$\boldsymbol{\sigma(q)}$	**x$_1$**	**x$_0$**
q_0	0	0
q_1	0	1
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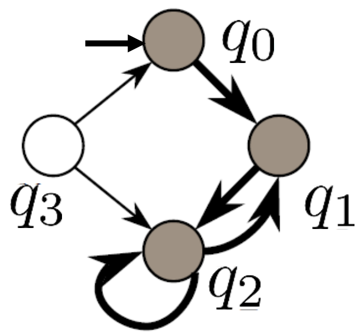
Compute reachable states:

$$\psi_{Q_{i+1}}(q') = \psi_{Q_i}(q') + (\,\exists q : \psi_{Q_i}(q) \cdot \psi_\delta(q, q'))$$

$$Q_3 = Q_2 \cup \{q_1, q_2\}$$
$$= \{q_0, q_1, q_2\}$$



$$\psi_{Q_2}(q') = \overline{x_1'} + \overline{x_0'}$$

$$\psi_{Q_3}(q') = \overline{x_1'} + \overline{x_0'} + (\,\underbrace{\exists q : (\overline{x_1} + \overline{x_0}) \cdot \psi_\delta(q, q'))}_{q_0, q_1, q_2}$$

$$= \overline{x_1'} + \overline{x_0'} + \overline{x_1'} + \overline{x_0'} = \overline{x_1'} + \overline{x_0'}$$

It's **always** a reachability problem

Or rather

The goal is to transform the problem at hand
to **encode it as a reachability problem**.

▶ Because these can be solved very efficiently

1. Work with sets of states
2. Use characteristic functions to represent sets of states
3. Use ROBDDs to encode characteristic functions

It's **always** a reachability problem

Or rather

The goal is to transform the problem at hand
to **encode it as a reachability problem**.

- Because these can be solved very efficiently

  1. Work with sets of states
  2. Use characteristic functions to represent sets of states
  3. Use ROBDDs to encode characteristic functions

- Comparison of finite automata

  1. Compute the set of jointly reachable states
  2. Compare the output values of two finite automata
  3. ...

Your turn to practice!
after the break

1. Familiarise yourself with the equivalence
   "set of states" $\equiv$ "characteristic functions"

2. Express system properties using
   characteristic functions

3. Draw and simplify BDDs to compare
   a specification and an implementation

Efficient state representation

- Set of states as Boolean function
- Binary Decision Diagram representation

Computing reachability

- Leverage efficient state representation
- Explore successor sets of states

Next week

Proving properties

- Temporal logic (CTL)
- Encoding as reachability problem