Slotted Aloha

(1)

Simplest possible randomized idea: Station transmits with prob p.

Prob. that my station succeeds:

$$P_1 = \underset{\substack{\uparrow \\ \text{I transmit}}}{p} \cdot \underset{\substack{\uparrow \\ \text{everybody else not.}}}{(1-p)^{n-1}}$$

Prob. that any station succeeds: $P = n \cdot P_1 = np \cdot (1-p)^{n-1}$

1. $\frac{d}{dx} f(x)g(x) = f'g + fg'$

2. $\frac{d}{dx} f(x)^k = k \cdot f' \cdot f^{k-1}$

Maximized for which p?

$$\frac{dP}{dp} = n(1-p)^{n-1} + np \cdot (n-1) \cdot (-1) \cdot (1-p)^{n-2}$$

$$= n(1-p)^{n-2}\big((1-p) - p(n-1)\big)$$

$$= \underbrace{n(1-p)^{n-2}}_{>0}(1-pn) \overset{!}{=} 0 \Rightarrow \underline{\underline{pn = 1}}$$

$$\Rightarrow P = n\tfrac{1}{n}\left(1-\tfrac{1}{n}\right)^{n-1} = \left(1-\tfrac{1}{n}\right)^{n-1} \geq 1/e \approx 36.8\%$$

$n = 2 \rightarrow P = 1/2$; $n = 3 \rightarrow P = \frac{4}{9}$; $n \rightarrow \infty$: $\frac{1}{e}$

- $n \geq 1$, $|t| < n$:

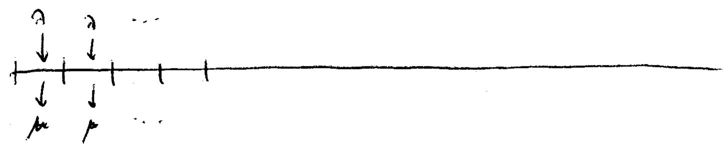
$$e^t\left(1-\frac{t^2}{n}\right) \leq \left(1+\frac{t}{n}\right)^n \leq e^t$$

- $k \geq 1$, $0 < p < 1$: $1-p \leq \left(1-\frac{p}{k}\right)^k$

Queing Theory

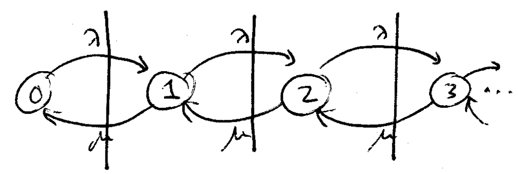
M/M/1 in each slot

Application wants to transmit a new packet with probability λ.
Mobile device manages to successfully send packet with prob. μ.



$$\rho = \frac{\lambda}{\mu} < 1 \qquad \text{(Mensa)}$$

N = # of jobs in system?



P_i = Prob. that system is in state i

$$P_0 \cdot \lambda = P_1 \cdot \mu \qquad \Leftrightarrow \quad P_1 = \frac{\lambda}{\mu} P_0 = \rho \cdot P_0$$

$$P_1 \cdot \lambda = P_2 \cdot \mu \qquad \rightarrow \quad P_2 = \rho \cdot P_1 = \rho^2 P_0$$

in general $P_i = \rho^i \cdot P_0$

$$\sum_{\forall i} P_i \overset{!}{=} 1 \quad \Rightarrow \quad P_0 \underbrace{\sum_{i=0}^{\infty} \rho^i}_{= \frac{1}{1-\rho}} = 1 \quad \Rightarrow \quad P_0 = \underline{\underline{1-\rho}}$$

$$E[N] = \sum_{i=0}^{\infty} i \cdot P_i = P_0 \underbrace{\sum_{i=0}^{\infty} i \cdot \rho^i}_{= \frac{\rho}{(1-\rho)^2}} = \underline{\underline{\frac{\rho}{1-\rho}}}$$

Little's Formula: $N = \lambda \cdot T \longrightarrow T = \frac{N}{\lambda} = \frac{1}{\mu(1-\rho)}$

Adaptive Slotted Aloha

New arrival with prob λ

n = # of stations in system

$\hat{n}$ = my estimate of n

$p = \frac{1}{\hat{n}}$

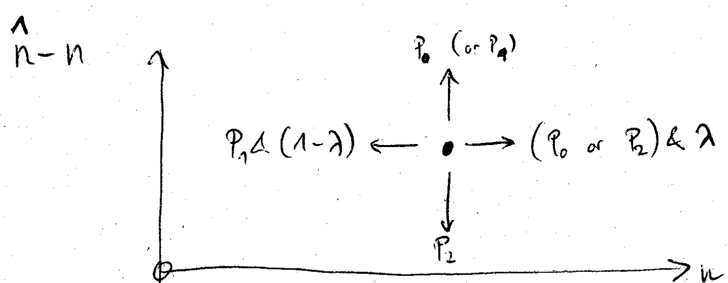
Prob. 0 send $= P_0 = (1-p)^n$

Prob. 1 send $= P_1 = p \cdot n \cdot (1-p)^{n-1}$

Prob. >1 send $= P_2 = 1 - P_0 - P_1$

Simplifications: $\left(1-\frac{1}{n}\right)^n \approx 1/e \Rightarrow$ for $\hat{n} \approx n$: $P_0 \approx 1/e$

$P_1 = pn\,(1-p)^{n-1} \approx 1 \cdot 1/e \approx 1/e \Rightarrow P_2 \approx 1 - 2/e$



$$dx = \lambda(1-P_1) - (1-\lambda)P_1$$

$$d\hat{n} = \begin{cases} P_0 \text{ (or } P_1) : & \hat{n} \leftarrow \hat{n} + \lambda - 1 \\ P_2 : & \hat{n} \leftarrow \hat{n} + \lambda + x \end{cases}$$

$$\Delta \quad \underbrace{P_0}_{1/e} \cdot 1 \overset{!}{=} \underbrace{P_2}_{\frac{e-2}{e}} \cdot x \quad \Rightarrow x = \frac{1}{e-2}$$

$$dy = (1-P_2)(\lambda-1) - P_2\left(\lambda + \frac{1}{e-2}\right)$$