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Principles of Distributed Computing
Exercise 4: Sample Solution

1 Bad Queues in a Mesh

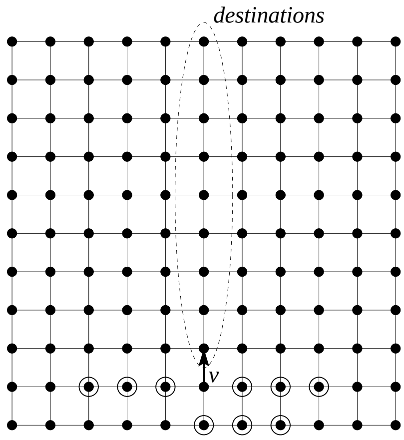

Figure 1: $m-2$ packets congest at node v

In order to obtain big queues at a node v, packets need to arrive from all three possible directions in each step. Therefore, the maximum number of destinations from one direction in a column is $m-2$. See Figure 1. In each step, the queue grows by 2 and there are $(m-2)/3$ steps. Thus the queue size grows to

$$\frac{2}{3}(m-2)$$

2 Good Queues in a Mesh

Following the lead given in the exercise we want to bound the probability P_{2em} that a particular column contains $2em$ or more destination packets. Analogous to the proof of Theorem 4.10 in the lecture, we have

$$P_{2em} < \binom{m^2}{2em} \cdot \left(\frac{1}{m}\right)^{2em} \tag{1}$$

(since we put $2em$ out of the m^2 destination packets in that column, each with a probability $1/m$). Using the inequality of the lecture (in the same proof) we can further simplify this to

$$P_{2em} < \left(\frac{em^2}{2em}\right)^{2em} \left(\frac{1}{m}\right)^{2em} = \left(\frac{1}{2}\right)^{2em} \tag{2}$$

to obtain that the probability for a single column to contain more than $2em$ packets is "really small" (i.e. in $o(2^{-m})$).

Since we want a bound on the column with the maximum number of destination packets, we can compute the probability P_{all} that all m columns contain *less* than $2em$ packets:

$$P_{\text{all}} = (1 - P_{2em})^m > \left(1 - \frac{1}{2^{2em}}\right)^m \tag{3}$$

To simplify things, we can use the following inequality

$$\left(1 - \frac{p}{k}\right)^k \geq 1 - p \tag{4}$$

for $0 < p < 1$ and $k \geq 1$. Plugging (4) into (3) we get

$$P_{\text{all}} > \left(1 - \frac{m2^{-2em}}{m}\right)^m \geq 1 - \frac{m}{2^{2em}} \geq 1 - \frac{1}{m} \tag{5}$$

where we used that $m/2^m \leq 1/m$.

Altogether, the argument is then as follows: The probability that all columns contain less than $O(m)$ packets is high, namely in $1 - O(1/m)$. Therefore, we also have a high probability that the column containing the most number of destinations also gets only $O(m)$ packets. To route a packet along a row takes at most $m - 1$ time steps. Once it has arrived at the designated column, it will have to wait for at most $O(m)$ other packets (with high probability). Altogether each packet needs time $O(m)$ to arrive at its destination.