Random Walks On Graphs

by Fabian Pichler 2004

a talk based on

L. Lovász, Random Walks On Graphs: A Survey, Combinatorics, Paul Erdös is Eighty, Vol. 2, Bolyai Society Mathematical Studies, Keszthely (Hungary), 2(1993), 1-46.

Overview

1 the model

2 applications and related papers

3 the abstract model (Markov chains)

4 some definitions and basic questions

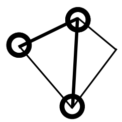
5 upper and lower bounds

6 Symmetry and access time

7 examples

1. the model

- A random sequence of connected nodes selected in the following way:



1 choose the start node

2 select a neighbour of it at random

3 move to it

4 finish walk or back to step 2.

2. applications

- a general model widely used
  - economics, share prices
  - physics, electrical networks, Brownian motion
  - mathematics, Laplace's equation
  - sampling problems, lattice points in a convex body

and related papers

- sampling in algorithm design
  (e.i. perfect matchings, volume of a convex body)
  [A. Broder, How hard is it to marry at random? (On the approximation of the permanent), Proc. 18th Annual ACM Symposium on Theory of Computing (1986), 50-58]
- software testing
  [H. Robinson, Graph Theory Techniques in Model-Based Testing, International Conference on Testing Computer Software 1999, Microsoft Corporation, (1999), http://www.geocities.com/harry_robinson_testing/graph_theory.htm]

related papers in distributed computing

- routing in circuit switching networks
  [A. Broder, A. Frieze and E. Upfal, Static and Dynamic Path Selection on Expander Graphs: A Random Walk Approach, STOC '97, El Paso, Texas USA]
- random walks with "back buttons"
  [R. Fagin, A. Karlin, J. Kleinberg et al., Random Walks with "Back Buttons", Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, Portland, Oregon, (May 2000), 484-493]
  (S. of Distributed Computing WS 03/04: Web as a Graph)

self-stabilising systems in distributed computing

- [1] token management
  [D. Coppersmith, P. Tetali and P. Winkler, Collisions among Random Walks on a Graph, SIAM Journal of Discrete Mathematics, Vol. 6, Issue 3 (August 1993), 363-374]
- [2] Group Communication in Ad-Hoc Networks
  [S. Dolev, E. Schiller and J. Welch, Random Walk for Self-Stabilizing Group Communication in Ad-Hoc Networks, Annual ACM Symposium on Principles of Distributed Computing, Proceedings of the twenty-first annual symposium on Principles of distributed computing, (2002), p. 259]

3. the abstract model

- Markov chain <—> random walk on directed graph with weighted edges
- time-reversible Markov chains <—> undirected graphs
- symmetric Markov chains <—> regular symmetric graphs
- for undirected graphs:
  We move with probability 1/degree(current node) to a given neighbour.

4. some definitions and basic questions

- I Does the random walk return to its starting point with probability one?
  Does it return infinitely often?
- I How long do we have to walk before we return to the starting point?
- I ...before we see a given node?
- I ...before we see all nodes?

I. Does the random walk return to its starting point with probability one? Does it return infinitely often?

- Pólya (1921) proved that:
  "If we do a random walk on a d-dimensional grid, then we return with probability 1 to the starting point infinitely often if d=2, BUT only a finite number of times if d≥3."

II. How long do we have to walk before we return to the starting point?

- Commute Time : the expected number of steps in a random walk starting at i, before node j is visited and then node i is reached again.
- $\kappa(i,j) = H(i,j) + H(j,i)$

III. ...before we see a given node?

- Access time or hitting time H_{ij} : the expected number of steps before node j is visited, starting from node i.

$$H(i,j) = \frac{1}{2}\left(k(i,j) + \sum_{u} p(u)\left[k(u,j) - k(u,i)\right]\right)$$

IV. ...before we see all nodes?

- Cover time: is the expected number of steps to reach every node.
- Upper and Lower Bounds: The cover time from any starting node in a graph with n nodes is at least (1-o(1))n log n and at most $(4/27 + o(1))n^3$.
  [U. Feige, A Tight Upper Bound on the Cover Time for Random Walks on Graphs, AND A Tight Lower Bound on the Cover Time for Random Walks on Graphs, IN Random Structures and Algorithms 6(1995), 51-54 AND 433-438]
  ---> used in [2]

5. Lower and Upper Bounds on access time

- Upper Bound:
  [G. Brightwell and P. Winkler, Maximum hitting time for random walks on graphs, J. Random Structures and Algorithms 1(1990), 263-276] --> used in [1]
  The access time between any two nodes of a graph on n nodes is at most
  - $(4/27)n^3 - (1/9)n^2 + (2/3)n - 1$ IF $n \equiv 0 \pmod 3$
  - $(4/27)n^3 - (1/9)n^2 + (2/3)n - (29/27)$ IF $n \equiv 1 \pmod 3$
  - $(4/27)n^3 - (1/9)n^2 + (4/9)n - (13/27)$ IF $n \equiv 2 \pmod 3$
- No non-trivial Lower Bound exists even for regular graphs!
  ---> proof on overhead projector

6. Symmetry and access time

- $H(i,j) \neq H(j,i)$ even on regular graphs
  ---> proof on overhead projetor
- [1]: For any three nodes u, v and w,
  $H(i,j) + H(j,k) + H(k,i) = H(i,k) + H(k,j) + H(j,i)$
  ---> proof on overhead projetor
  - Lemma: The nodes of any graph can be ordered so that if i precedes j then $H(i,j) \leq H(j,i)$. Such an ordering can be obtained by fixing any node t, and order the nodes according to the value of $H(i,t) - H(t,i)$.

7. Examples

- Example 1: Let us determine the access time for a complete graph on n nodes {0,...,n−1}

$$H(0,1) = \sum_{i=1}^{\infty} i \left(\frac{n-2}{n-1}\right)^{i-1} \frac{1}{n-1} = n-1$$

- Example 2: Let us determine the cover time for a complete graph on n nodes {0,...,n−1}

$$E(\tau_n) = \sum_{i=1}^{n-1} E(\tau_{i+1} - \tau_i) = \sum_{i=1}^{n-1} \frac{n-1}{n-i} \approx n \log n$$

- Example 3: A graph with particularly bad random walk properties is obtained by taking a clique of size n/2 and attach to it an endpoint of a path of length n/2. Why?

Questions ?

Thank you for your attention.