

Discrete Event Systems

Exercise 7: Sample Solution

1 An Unsolvable Problem

- a) It is surprisingly easy to prove that your boss is demanding too much. Assume a function $halt(P : Program) : boolean$ which takes a program P as a parameter and returns a boolean value denoting whether P terminates or not.

Now consider the following program X which calls the $halt()$ function with itself as an argument just to do the contrary:

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L: if halt(X) goto L else exit;
```

Obviously, if $halt(X)$ is true X will loop forever, and vice versa.

- b) If the simulation stops we can definitively decide that the program does not contain an endless loop. However, while the simulation is still running, we do not know whether it will finish in the next two seconds or run forever. Put differently: There is no upper bound on the execution time of the simulation after which we can be sure that the program contains an endless loop.
- c) As we have seen, it is not possible to predict whether a general program terminates or not. However, under certain constraints we can solve the halting problem all the same. For example, consider a restricted language with only one form of a loop (no recursion etc.):

```
for(init, end, inc){...}
```

where $init$, end and inc are constants in \mathbb{Z} . The loop starts with the value $init$ and adds inc to $init$ in every round until this sum exceeds end if $end > 0$ or until it falls below end if $end < 0$. Obviously, there is a simple way to decide whether a program written in this language terminates: For every loop, we check whether $sgn(inc) = sgn(end)$, where $sgn(\cdot)$ is the algebraic sign. If not, the program contains an endless loop (unless $init$ itself already fulfills the termination criterion which is also easy to verify).

2 Dolce Vita in Rome

Consider the following indicator variable for shop i : X_i is 1 if Hector and Rachel buy ice cream at the i -th shop and 0 otherwise. Since the probability that the i -th shop is the best so far equals $1/i$, we have $E[X_i] = 1/i \cdot 1 = 1/i$.

The total number of ice creams can be expressed by

$$X := X_1 + X_2 + \dots + X_n.$$

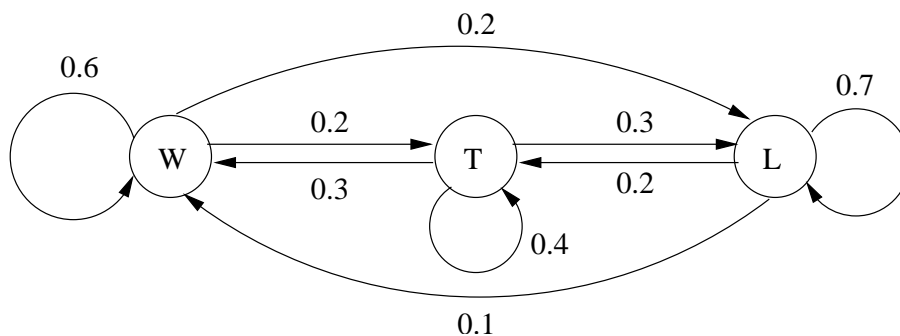
Therefore

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n 1/i = H_n.$$

Thereby, H_n is the so-called *Harmonic Number*, which is roughly $H_n = \ln(n) + O(1)$. Thus, the two students roughly consume a logarithmic number of ice creams (in the total number of shops).

3 Soccer Betting

a) The following Markov chain models the different transition probabilities (W :Win, T :Tie, L :Loss):



b) The transition matrix P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{pmatrix} \quad (1)$$

Since the FCB has lost its previous game, the Markov chain is currently in the state L and hence, the initial vector is $q_0 = (0, 0, 1)$. The probability distribution q_3 for the game against the Grasshoppers is therefore given by

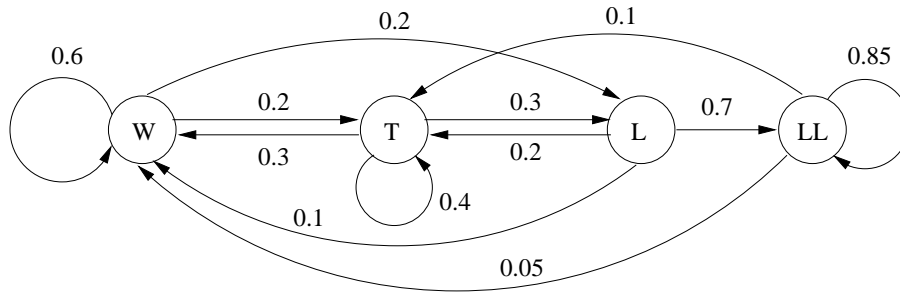
$$q_3 = q_0 \cdot P^3 = (0, 0, 1) \cdot \begin{pmatrix} 0.368 & 0.248 & 0.384 \\ 0.321 & 0.256 & 0.423 \\ 0.243 & 0.248 & 0.509 \end{pmatrix} = (0.243, 0.248, 0.509). \quad (2)$$

Given the quotas of the exercise, the expected gains for each of the three possibilities (W , T , L) is

$$\begin{aligned} E[W] &= 0.243 \cdot 3.5 = 0.8505 \\ E[T] &= 0.248 \cdot 3.5 = 0.868 \\ E[L] &= 0.509 \cdot 2 = 1.018. \end{aligned}$$

Therefore, the best choice is to bet on a loss.

- c) The new Markov chain model looks like this. In addition to the three states W , T , and L , there is now a new state LL which is reached if the team has lost twice in a row. The new transition matrix



P is

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.2 & 0 \\ 0.3 & 0.4 & 0.3 & 0 \\ 0.1 & 0.2 & 0 & 0.7 \\ 0.05 & 0.1 & 0 & 0.85 \end{pmatrix} \quad (3)$$

As the FCB has (in this order) won and lost its last two games, the Markov chain is currently in the state $q_0 = (0, 0, 1, 0)$. The probabilities for the game against the Grasshoppers can again be computed as

$$\begin{aligned} q_3 &= q_0 \cdot P^3 = (0, 0, 1, 0) \cdot \begin{pmatrix} 0.361 & 0.234 & 0.16 & 0.245 \\ 0.3105 & 0.235 & 0.15 & 0.3045 \\ 0.18175 & 0.1745 & 0.082 & 0.56175 \\ 0.42125 & 0.15475 & 0.061 & 0.6421 \end{pmatrix} \\ &= (0.18175, 0.1745, 0.082, 0.5617). \end{aligned}$$

Finally, we can compute the expected win for each of the three possible bets:

$$\begin{aligned} E[W] &= 0.18175 \cdot 3.5 = 0.636125 \\ E[T] &= 0.1745 \cdot 3.5 = 0.61 \\ E[L] &= (0.082 + 0.5617) \cdot 2 = 1.2874. \end{aligned}$$

Clearly, the addition of the state LL worsens the situation for FCB.